ARRANGEMENT GROUPS, LOWER CENTRAL SERIES, AND MASSEY PRODUCTS

Alex Suciu

Northeastern University

Workshop on Hyperplane Arrangements

Institute of Mathematics
Vietnam Academy of Science and Technology

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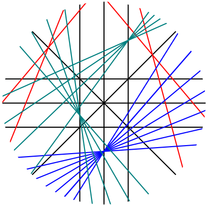


LOWER CENTRAL SERIES

- Let G be a group. The *lower central series* $\{\gamma_k(G)\}_{k\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \lhd G$, then $[H, K] \lhd G$.
- The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of G. Moreover $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- $\gamma_2(G) = [G, G]$ is the derived subgroup, and so $G/\gamma_2(G) = G_{\mathrm{ab}}$.
- $[\gamma_k(G), \gamma_k(G)] \lhd \gamma_{k+1}(G)$, and thus the LCS quotients,

  $$\mathrm{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  are abelian.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \mathrm{gr}_k(G)$.

ASSOCIATED GRADED LIE ALGEBRA

- Fix a coefficient ring $\Bbbk$. Given a group G, we let

$$\operatorname{gr}(G,\Bbbk) = \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)\otimes \Bbbk.$$

- This is a graded Lie algebra, with Lie bracket $[\ ,\ ]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.
- For $\Bbbk = \mathbb{Z}$, we simply write $\operatorname{gr}(G) = \operatorname{gr}(G,\mathbb{Z})$.
- The construction is functorial.
- Example: if F_n is the free group of rank n, then
  - $\operatorname{gr}(F_n)$ is the free Lie algebra $\operatorname{Lie}(\mathbb{Z}^n)$.
  - $\operatorname{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k}\sum_{d\mid k}\mu(d)n^{\frac{k}{d}}$.

CHEN LIE ALGEBRAS

- Let $G^{(i)}$ be the *derived series* of G, starting at $G^{(1)} = G'$, $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.
- The quotient groups, $G/G^{(i)}$, are solvable; $G/G' = G_{\text{ab}}$, while G/G'' is the maximal metabelian quotient of G.
- The *i-th Chen Lie algebra* of G is defined as $\operatorname{gr}(G/G^{(i)}, \Bbbk)$. Clearly, this construction is functorial.
- The projection $q_i \colon G \twoheadrightarrow G/G^{(i)}$, induces a surjection $\operatorname{gr}_k(G; \Bbbk) \twoheadrightarrow \operatorname{gr}_k(G/G^{(i)}; \Bbbk)$, which is an iso for $k \leqslant 2^i - 1$.
- Assuming G is finitely generated, write $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- Example (K.-T. Chen 1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geqslant 2$.

HOLONOMY LIE ALGEBRA

- A quadratic approximation of the Lie algebra $\operatorname{gr}(G, \Bbbk)$, where $\Bbbk$ is a field, is the *holonomy Lie algebra* of G, which is defined as

$$\mathfrak{h}(G, \Bbbk) := \operatorname{Lie}(H_1(G, \Bbbk))/\langle \operatorname{im}(\mu_G^\vee) \rangle,$$

  where
  - $L = \operatorname{Lie}(V)$ the free Lie algebra on the $\Bbbk$-vector space $V = H_1(G; \Bbbk)$, with $L_1 = V$ and $L_2 = V \wedge V$.
  - $\mu_G^\vee \colon H_2(G, \Bbbk) \to V \wedge V$ is the dual of the cup product map $\mu_G \colon H^1(G; \Bbbk) \wedge H^1(G; \Bbbk) \to H^2(G; \Bbbk)$.

- There is a surjective morphism of graded Lie algebras,

$$\mathfrak{h}(G, \Bbbk) \twoheadrightarrow \operatorname{gr}(G; \Bbbk) \ , \qquad (*)$$

  which restricts to isomorphisms $\mathfrak{h}_k(G, \Bbbk) \to \operatorname{gr}_k(G; \Bbbk)$ for $k \leqslant 2$.

ARRANGEMENT GROUPS AND LIE ALGEBRAS

- Let $\mathcal{A} = \{\ell_1, \dots, \ell_n\}$ be an affine line arrangement in $\mathbb{C}^2$, and let $G = G(\mathcal{A})$ be the fundamental group of the complement of $\mathcal{A}$.
- The holonomy Lie algebra $\mathfrak{h}(\mathcal{A}) := \mathfrak{h}(G(\mathcal{A}))$ has (combinatorially determined) presentation

$$\mathfrak{h}(\mathcal{A}) = \Big\langle x_1, \dots, x_n \mid \sum_{k \in P} [x_j, x_k],\ j \in \widehat{P},\ P \in \mathcal{P} \Big\rangle$$

  where x_i represents the meridian about the i-th line, $\mathcal{P} \subset 2^{[n]}$ is the set of multiple points, and $\widehat{P} = P \backslash \{\max P\}$ for $P \in \mathcal{P}$.
- Thus, every double point $P = L_i \cap L_j$ contributes a relation $[x_i, x_j]$, each triple point $P = L_i \cap L_j \cap L_k$ contributes two relations, $[x_i, x_j] + [x_i, x_k]$ and $-[x_i, x_j] + [x_j, x_k]$, etc.
- Consequently, $\mathfrak{h}_1(\mathcal{A})$ is free abelian with basis $\{x_1, \dots, x_n\}$, while $\mathfrak{h}_2(\mathcal{A})$ is free abelian of rank $\phi_2 = \binom{n}{2} - \sum_{P \in \mathcal{P}} (|P| - 1)$, with basis $\{[x_i, x_j] :\ i, j \in \widehat{P},\ P \in \mathcal{P}\}$.

- The canonical projection $\mathfrak{h}(G,\mathbb{Q}) \twoheadrightarrow \operatorname{gr}(G,\mathbb{Q})$ is an isomorphism. Thus, the LCS ranks $\phi_k(G)$ are combinatorially determined.
- (Falk–Randell 1985) If $\mathcal{A}$ is *supersolvable*, with exponents $d_1,\dots,d_\ell$, then $G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$ (almost direct product) and
$$\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i}).$$
- (Papadima–S. 2006) If $\mathcal{A}$ is *decomposable*, then $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism, and $\operatorname{gr}_k(G)$ is free abelian of rank
$$\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)}) \text{ for } k \geqslant 2.$$
- (S. 2001) For $G = G(\mathcal{A})$, the groups $\operatorname{gr}_k(G)$ may have non-zero torsion. Question: Is that torsion combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2018): Answer: No!

Malcev Lie algebra

- Let $\Bbbk$ be a field of characteristic 0. The group-algebra $\Bbbk G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon \colon \Bbbk G \to \Bbbk$.
- Let $I = \ker \varepsilon$. The I-adic completion $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G/I^k$ is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements,

  $$\mathfrak{m}(G, \Bbbk) = \operatorname{Prim}(\widehat{\Bbbk G}),$$

  with bracket $[x, y] = xy - yx$, is a complete, filtered Lie algebra, called the *Malcev Lie algebra* of G.
- If G is finitely generated, then $\mathfrak{m}(G, \Bbbk) = \varprojlim_k \mathcal{L}(G/\gamma_k(G) \otimes \Bbbk)$, and

  $$\operatorname{gr}(\mathfrak{m}(G, \Bbbk)) \cong \operatorname{gr}(G, \Bbbk).$$

Formality and filtered formality

- Let G be a finitely generated group, $\Bbbk$ a field of characteristic 0.
- G is *filtered-formal* (over $\Bbbk$), if there is an isomorphism of filtered Lie algebras,

$$\mathfrak{m}(G;\Bbbk) \cong \widehat{\operatorname{gr}}(G;\Bbbk).$$

- G is 1*-formal* (over $\Bbbk$) if it is filtered formal and the canonical projection $\mathfrak{h}(G,\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism; that is,

$$\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathfrak{h}}(G;\Bbbk).$$

- An obstruction to 1-formality is provided by the Massey products $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H^2(G,\Bbbk)$, for $\alpha_i \in H^1(G,\Bbbk)$ with $\alpha_1\alpha_2 = \alpha_2\alpha_3 = 0$.

Theorem (S.–Wang)

The above formality properties are preserved under finite direct products and coproducts, split injections, passing to solvable quotients, as well as extension or restriction of coefficient fields.

- Examples of 1-formal groups
  - Fundamental groups of compact Kähler manifolds; e.g., surface groups.
  - Fundamental groups of complements of complex algebraic affine hypersurfaces; e.g., arrangement groups, free groups.
  - Right-angled Artin groups.
- Examples of filtered formal groups
  - Finitely generated, torsion-free, 2-step nilpotent groups with torsion-free abelianization; e.g., the Heisenberg group.
  - Fundamental groups of Sasakian manifolds.
  - Fundamental groups of graphic configuration spaces of surfaces of genus $g \geqslant 1$; e.g., pure braid groups of elliptic curves.
- Examples of non-filtered formal groups
  - Certain finitely generated, torsion-free, 3-step nilpotent groups.

CHEN LIE ALGEBRAS AND FILTERED FORMALITY

THEOREM (PAPADIMA–S., S.–WANG)

For each $i \geqslant 2$, there is an isomorphism of complete, separated, filtered Lie algebras,

$$\mathfrak{m}(G/G^{(i)}; \Bbbk) \cong \mathfrak{m}(G; \Bbbk)/\overline{\mathfrak{m}(G; \Bbbk)^{(i)}}.$$

THEOREM (SW)

For each $i \geqslant 2$, the quotient map $G \twoheadrightarrow G/G^{(i)}$ induces a natural epimorphism of graded $\Bbbk$-Lie algebras,

$$\operatorname{gr}(G; \Bbbk)/\operatorname{gr}(G; \Bbbk)^{(i)} \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G^{(i)}; \Bbbk)\ .$$

Moreover, if G is filtered formal, this map is an isomorphism and $G/G^{(i)}$ is also filtered formal.

The map $\mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ induces $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)})$.

Corollary (Papadima–S. 2004)

If G is 1-formal, then $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G,\Bbbk)^{(i)} \xrightarrow{\simeq} \operatorname{gr}(G/G^{(i)},\Bbbk)$.

Theorem

Let G_1 and G_2 be two $\Bbbk$-filtered formal groups. Then every morphism of graded Lie algebras, $\alpha\colon \operatorname{gr}(G_1;\Bbbk) \to \operatorname{gr}(G_2,\Bbbk)$, induces a morphism $\alpha_i\colon \operatorname{gr}(G_1/G_1^{(i)};\Bbbk) \to \operatorname{gr}(G_2/G_2^{(i)};\Bbbk)$, for each $i \geqslant 1$. Consequently,

$$\operatorname{gr}(G_1;\Bbbk) \cong \operatorname{gr}(G_2;\Bbbk) \implies \operatorname{gr}(G_1/G_1^{(i)};\Bbbk) \cong \operatorname{gr}(G_2/G_2^{(i)};\Bbbk).$$

Taking $i = 2$, we obtain:

Corollary

If G_1 and G_2 are $\Bbbk$-filtered formal and $\theta_k(G_1) \neq \theta_k(G_2)$ for some $k \geqslant 1$, then $\operatorname{gr}(G_1,\Bbbk) \not\cong \operatorname{gr}(G_2,\Bbbk)$, as graded Lie algebras.

PURE BRAID GROUPS AND THEIR FRIENDS

- Consider the groups
  - $P_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}))$—the pure braid group on n strings.
  - $P\Sigma_n^+$—the upper McCool group.
  - $\Pi_n = \prod_{i=1}^{n-1} F_i$.
- For each $n \geqslant 1$, they have the same LCS ranks and Betti numbers.
- For each $n \leqslant 3$, they are pairwise isomorphic.

PROPOSITION (SW)

For each $n \geqslant 4$, the graded Lie algebras $\mathrm{gr}(P_n, \mathbb{Q})$, $\mathrm{gr}(P\Sigma_n^+, \mathbb{Q})$, and $\mathrm{gr}(\Pi_n, \mathbb{Q})$ are pairwise non-isomorphic.

Follows from previous corollary (with, say, $k = 4$), and:

- All these groups are 1-formal (Brieskorn/Berceanu–Papadima/—).
- $\theta_k(P_n) = (k-1)\binom{n+1}{4}$ for $k \geqslant 3$. [Cohen–S.]
- $\theta_k(P\Sigma_n^+) = \binom{n+1}{4} + \sum_{i=3}^{k} \binom{n+i-2}{i+1}$ for $k \geqslant 3$. [S.–Wang]
- $\theta_k(\Pi_n) = (k-1)\binom{k+n-2}{k+1}$ for $k \geqslant 2$. [Chen, CS]

Nilpotent quotients

- Consider the tower of nilpotent quotients of a group G,

$$\cdots \longrightarrow G/\gamma_4(G) \xrightarrow{q_3} G/\gamma_3(G) \xrightarrow{q_2} G/\gamma_2(G)\,.$$

- We then have central extensions

$$0 \longrightarrow \operatorname{gr}_k(G) \longrightarrow G/\gamma_{k+1}(G) \xrightarrow{q_k} G/\gamma_k(G) \longrightarrow 0\,.$$

- Passing to classifying spaces, we obtain commutative diagrams,

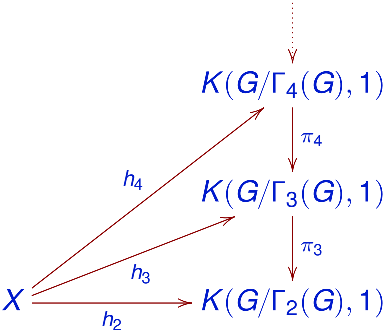
$$\begin{array}{ccc} & & K(G/\gamma_{k+1}(G),1) \\ & \overset{\psi_{k+1}}{\nearrow} & \downarrow{\scriptstyle \pi_k} \\ G & \xrightarrow[\psi_k]{} & K(G/\gamma_k(G),1) \end{array}$$

- The map π_k may be viewed as the fibration with fiber $K(\operatorname{gr}_k(G),1)$ obtained as the pullback of the path space fibration with base $K(\operatorname{gr}_k(G),2)$ via a k-invariant $\chi_k\colon K(G/\gamma_k(G),1) \to K(\operatorname{gr}_k(G),2)$.

- Let X be a connected CW-complex, and let $G = \pi_1(X)$.
- A $K(G,1)$ can be constructed by adding to X cells of dimension 3 or higher. Thus, $H_2(G,\mathbb{Z})$ is a quotient of $H_2(X,\mathbb{Z})$.
- Let $\iota\colon X \to K(G,1)$ be the inclusion, and let

$$h_k = \psi_k \circ \iota\colon X \to K(G/\gamma_k(G),1).$$

- We obtain a Postnikov tower of fibrations,



INJECTIVE HOLONOMY AND k-INVARIANTS

- As noted by Stallings, there is an exact sequence,

$$H_2(X;\mathbb{Z}) \xrightarrow{(h_k)_*} H_2(G/\gamma_k(G);\mathbb{Z}) \xrightarrow{\chi_k} \operatorname{gr}_k(G) \longrightarrow 0\ .$$

  In general, this sequence is natural but not split exact.

- The homomorphism

$$(h_2)_*\colon H_2(X;\mathbb{Z}) \longrightarrow H_2(G/\gamma_2(G);\mathbb{Z}) \cong H_1(G;\mathbb{Z}) \wedge H_1(G;\mathbb{Z})$$

  is the *holonomy map* of X (over $\mathbb{Z}$).

- When $H_1(G;\mathbb{Z})$ is torsion-free, set

$$\mathfrak{h}(G) = \operatorname{Lie}(H_1(G;\mathbb{Z}))/\langle \operatorname{im}((h_2)_*)\rangle.$$

- As before, get surjective morphism $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$, which is injective in degrees $k \leqslant 2$.

Suppose $H = H_1(G;\mathbb{Z})$ is a finitely-generated, free abelian group, and the map $(h_2)_*\colon H_2(G;\mathbb{Z}) \to H \wedge H$ is injective.

THEOREM (RYBNIKOV, PORTER–S.)

The canonical projection $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism.

THEOREM (PORTER–S.)

For each $k \geqslant 3$, there is a split exact sequence,

$$0 \longrightarrow \mathrm{gr}_k(G) \xrightarrow{\ i\ } H_2(G/\gamma_k(G);\mathbb{Z}) \xrightarrow{\ \pi\ } H_2(X;\mathbb{Z}) \longrightarrow 0 . \qquad (\dagger)$$

with splitting $\sigma\colon H_2(G/\gamma_k(G);\mathbb{Z}) \to \mathrm{gr}_k(G)$.

Moreover, the k-invariant of the extension from $G/\gamma_k(G)$ to $G/\gamma_{k+1}(G)$,

$$\chi_k \in \mathrm{Hom}(H_2(G/\gamma_k(G)), \mathrm{gr}_k(G)),$$

with respect to the direct sum decomposition defined by σ, is given by $\chi_k(x, c) = x - \lambda(c)$, where $\lambda = \sigma \circ (h_k)_*$.

A homological version of Rybnikov's theorem

- Let X_a and X_b be two path-connected spaces with
  - Finitely generated, torsion-free H_1.
  - Injective holonomy map $H_2 \to H_1 \wedge H_1$.
- Let G_a and G_b be the respective fundamental groups.
- A homomorphism $f\colon G_a \to G_b$ induces homomorphisms on nilpotent quotients, $f_k\colon G_a/\gamma_k(G_a) \to G_b/\gamma_k(G_b)$.
- Suppose there is an isomorphism of graded algebras,

$$g\colon H^{\leqslant 2}(X_b) \to H^{\leqslant 2}(X_a).$$

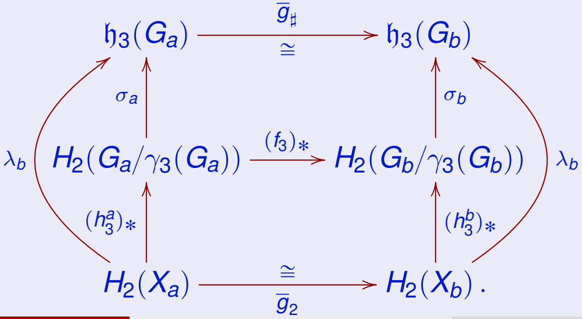
  Set $\overline{g} = g^\vee\colon H_{\leqslant 2}(X_a) \to H_{\leqslant 2}(X_b)$.
- There is then an isomorphism $G_a/\gamma_3(G_a) \xrightarrow{\simeq} G_b/\gamma_3(G_b)$.
- Moreover, the isomorphism $\overline{g}_1\colon H_1(X_a) \to H_1(X_b)$ induces an isomorphism $\overline{g}_\sharp\colon \mathfrak{h}_3(G_a) \to \mathfrak{h}_3(G_b)$.

THEOREM (RYBNIKOV, PORTER–S.)

Let $\sigma_b \colon H_2(G_b/\Gamma_3(G_b)) \to \mathfrak{h}_3(G_b)$ *be any left splitting of* $(\dagger)$*, and let* $f_3 \colon G_a/\gamma_3(G_a) \xrightarrow{\cong} G_b/\gamma_3(G_b)$ *be any extension of* $\overline{g}$*. Then* f_3 *extends to an isomorphism*

$$f_4 \colon G_a/\gamma_4(G_a) \xrightarrow{\cong} G_b/\gamma_4(G_b)$$

if and only if there are liftings $h_3^c \colon X_c \to K(G_c/\gamma_3(G_c), 1)$ *for* $c = a$ *and* b *such that the following diagram commutes*



An extension to characteristic p

- Let $p = 0$ or a prime.
- Given a group G, define subgroups $\gamma_k^p(G)$ as $\gamma_1^p(G) = G$ and

$$\gamma_{k+1}^p(G) = \langle gug^{-1}u^{-1}v^p : g \in G,\ u, v \in \gamma_k^p(G) \rangle.$$

- $\{\gamma_k^p(G)\}_{k \geqslant 1}$ is a descending central series of normal subgroups.
- For $p = 0$ it is the LCS; for $p \neq 0$ it is the most rapidly descending central series whose successive quotients are $\mathbb{Z}_p$-vector spaces.
- All the above results work for $p > 0$, by replacing $\gamma_k(G) \rightsquigarrow \gamma_k^p(G)$, $\mathfrak{h}_k(G) \rightsquigarrow \mathfrak{h}_k(G, \mathbb{Z}_p)$, and $H_*(-, \mathbb{Z}) \rightsquigarrow H_*(-, \mathbb{Z}_p)$.
- The entries of the matrices λ_a and λ_b are generalized Massey triple products in $H^2(X_b, \mathbb{Z}_p)$ and $H^2(X_a, \mathbb{Z}_p)$, respectively.

RYBNIKOV'S ARRANGEMENTS

- For groups of hyperplane arrangements, $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are torsion free. Moreover, the holonomy map is injective, and so $\mathfrak{h}_3 \cong \operatorname{gr}_3$.
- The obstruction to extending $\overline{g}$ to an isomorphism from $G/\gamma_4(G_a)$ to $G/\gamma_4(G_b)$ is computed by generalized Massey triple products.
- Rybnikov used the above theorem (with $n = 3$) to show that arrangement groups are not combinatorially determined.
- Starting from a realization $\mathcal{A}$ of the MacLane matroid over $\mathbb{C}$, he constructed a pair of arrangements of 13 planes in $\mathbb{C}^3$, $\mathcal{A}^+$ and $\mathcal{A}^-$, such that
  - $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, and thus $G^+/\gamma_3(G^+) \cong G^-/\gamma_3(G^-)$.
  - $G^+/\gamma_4(G^+) \not\cong G^-/\gamma_4(G^-)$.
- Goal: Make explicit the generalized Massey products (over $\mathbb{Z}_3$) that distinguish these two nilpotent quotients.

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