Poincaré duality and resonance varieties

doi:10.1017/prm.2019.55 · arxiv:1809.01801

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GASC Seminar

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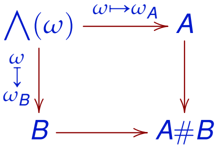
November 25, 2019

POINCARÉ DUALITY ALGEBRAS

- Let A be a graded, graded-commutative algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i\geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - $\cdot\colon A^i \otimes_{\Bbbk} A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in A^j$.
- We will assume that A is connected ($A^0 = \Bbbk \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\Bbbk} A^i$ are finite).
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- That is, A is a graded, graded-commutative Gorenstein Artin algebra of socle degree m.

- If A is a PD_m algebra, then:
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
- Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
- The *orientation class* is $\omega_A := 1^\vee$.
- We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.
- The class of $\Bbbk$-PD algebras is closed under taking tensor products and connected sums.
  - If A is PD_m and B is PD_n, then $A \otimes_{\Bbbk} B$ is PD_{m+n}.
  - If A and B are PD_m, then $A\#B$ is PD_m, where



THE ASSOCIATED ALTERNATING FORM

- Associated to a $\mathbb{k}$-PD_m algebra there is an alternating m-form,

$$\mu_A\colon \bigwedge^m A^1 \to \mathbb{k}, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$

- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the dual basis for A^2.
- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

  where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Let $A_i = (A^i)^*$. We may then view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

  which encodes the algebra structure of A.
- For instance, $\mu_{A\#B} = \mu_A + \mu_B$.

CLASSIFICATION OF ALTERNATING FORMS

(Following J. Schouten, G. Gurevich, D. Djoković, A. Cohen–A. Helminck, ...)

- Let V be a $\Bbbk$-vector space of dimension n. The group $\mathrm{GL}(V)$ acts on $\bigwedge^m(V^*)$ by $(g\cdot\mu)(a_1\wedge\cdots\wedge a_m)=\mu\left(g^{-1}a_1\wedge\cdots\wedge g^{-1}a_m\right)$.
- The orbits of this action are the equivalence classes of alternating m-forms on V. (We write $\mu\sim\mu'$ if $\mu'=g\cdot\mu$.)
- Over $\bar{\Bbbk}$, the Zariski closures of these orbits define affine algebraic varieties.
- There are finitely many orbits over $\bar{\Bbbk}$ only if $n^2\geqslant\binom{n}{m}$, that is, $m\leqslant 2$ or $m=3$ and $n\leqslant 8$.
- For $\bar{\Bbbk}=\mathbb{C}$, each complex orbit has only finitely many real forms.
- When $m=3$, and $n=8$, there are 23 complex orbits, which split into either 1, 2, or 3 real orbits, for a total of 35 real orbits.

- Let A and B be two PD_m algebras. We say that a morphism of graded algebras $\varphi\colon A \to B$ has *non-zero degree* if the linear map $\varphi^m\colon A^m \to B^m$ is non-zero. (Equivalently, φ is injective.)
- A and B are isomorphic as PD_m algebras if and only if they are isomorphic as graded algebras, in which case $\mu_A \sim \mu_B$.

PROPOSITION

For two PD_3 algebras A and B, the following are equivalent.

1. $A \cong B$, *as PD_3 algebras.*
2. $A \cong B$, *as graded algebras.*
3. $\mu_A \sim \mu_B$.

- We thus have a bijection between isomorphism classes of 3-dimensional Poincaré duality algebras and equivalence classes of alternating 3-forms, given by $A \leftrightsquigarrow \mu_A$.

POINCARÉ DUALITY IN ORIENTABLE MANIFOLDS

- If M is a compact, connected, orientable, m-dimensional manifold, then the cohomology ring $A = H^{\boldsymbol{\cdot}}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."

- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

RESONANCE VARIETIES

- Let A be a graded, graded-commutative, connected, locally finite algebra over a field $\Bbbk$ (with $\operatorname{char}\Bbbk \neq 2$).
- For each $a \in A^1$ we have $a^2 = -a^2$, and so $a^2 = 0$. We then obtain a cochain complex of $\Bbbk$-vector spaces,

$$(A, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.
- The *resonance varieties* of A (in degree $i \geqslant 0$ and depth $k \geqslant 0$):

$$\mathcal{R}_k^i(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \delta_a) \geqslant k\}.$$

- An element $a \in A^1$ belongs to $\mathcal{R}_k^i(A)$ if and only if there exist $u_1, \dots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in A^{i+1}, and the set $\{au, u_1, \dots, u_k\}$ is linearly independent in A^i, for all $u \in A^{i-1}$.

- Set $b_j = b_j(A)$. For each $i \geqslant 0$, we have a descending filtration,

$$A^1 = \mathcal{R}^i_0(A) \supseteq \mathcal{R}^i_1(A) \supseteq \cdots \supseteq \mathcal{R}^i_{b_i}(A) = \{0\} \supset \mathcal{R}^i_{b_{i+1}}(A) = \emptyset.$$

- A linear subspace $U \subset A^1$ is *isotropic* if the restriction of $\cdot\colon A^1 \wedge A^1 \to A^2$ to $U \wedge U$ is the zero map (i.e., $ab = 0$, $\forall a, b \in U$).
- If $U \subseteq A^1$ is an isotropic subspace of dimension k, then $U \subseteq \mathcal{R}^1_{k-1}(A)$.
- $\mathcal{R}^1_1(A)$ is the union of all isotropic planes in A^1.
- If $\Bbbk \subset \mathbb{K}$ is a field extension, then the $\Bbbk$-points on $\mathcal{R}^i_k(A \otimes_{\Bbbk} \mathbb{K})$ coincide with $\mathcal{R}^i_k(A)$.
- Let $\varphi\colon A \to B$ be a morphism of graded, connected algebras. If the map $\varphi^1\colon A^1 \to B^1$ is injective, then $\varphi^1(\mathcal{R}^1_k(A)) \subseteq \mathcal{R}^1_k(B)$, $\forall k$.

RESONANCE AND THE BGG CORRESPONDENCE

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.
- The Bernstein–Gelfand–Gelfand correspondence yields a cochain complex of finitely generated, free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i_A} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}_A} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i_A(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.
- The specialization of $(A \otimes S, \delta)$ at $a \in A^1$ coincides with (A, δ_a), that is, $\delta^i_A|_{x_j = a_j} = \delta^i_a$.

- By definition, an element $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if

$$\operatorname{rank} \delta^{i-1}_a + \operatorname{rank} \delta^i_a \leqslant b_i(A) - k.$$

- Let $I_r(\psi)$ denote the ideal of $r \times r$ minors of a $p \times q$ matrix ψ with entries in S, where $I_0(\psi) = S$ and $I_r(\psi) = 0$ if $r > \min(p, q)$. Then:

$$\begin{aligned}\mathcal{R}^i_k(A) &= V\Big(I_{b_i(A)-k+1}\big(\delta^{i-1}_A \oplus \delta^i_A\big)\Big)\\ &= \bigcap_{s+t=b_i(A)-k+1} \Big(V\big(I_s(\delta^{i-1}_A)\big) \cup V\big(I_t(\delta^i_A)\big)\Big).\end{aligned}$$

- In particular, $\mathcal{R}^1_k(A) = V(I_{n-k}(\delta^1_A))$ $(0 \leqslant k < n)$ and $\mathcal{R}^1_n(A) = \{0\}$.
- The (degree i, depth k) *resonance scheme* $\boldsymbol{\mathcal{R}}^i_k(A)$ is defined by the ideal $I_{b_i(A)-k+1}\big(\delta^{i-1}_A \oplus \delta^i_A\big)$; its underlying set is $\mathcal{R}^i_k(A)$.

EXAMPLE (EXTERIOR ALGEBRA)

Let $E = \bigwedge V$, where $V = \Bbbk^n$, and $S = \mathrm{Sym}(V)$. Then $\mathbf{L}(E)$ is the Koszul complex on V. E.g., for $n = 3$:

$$S \xrightarrow{\begin{pmatrix} x_3 & -x_2 & x_1 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_2 & -x_1 & 0 \\ x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S \,.$$

This chain complex provides a free resolution $\mathbf{L}(E) \to \Bbbk$ of the trivial S-module $\Bbbk$. Hence,

$$\mathcal{R}^i_k(E) = \begin{cases} \{0\} & \text{if } k \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example (Non-zero resonance)

Let $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2\rangle$, and set $S = \Bbbk[x_1, x_2, x_3]$. Then

$$\mathbf{L}(A): \quad S^2 \xrightarrow{\begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_k(A) = \begin{cases} \{x_3 = 0\} & \text{if } k = 1, \\ \{0\} & \text{if } k = 2 \text{ or } 3, \\ \emptyset & \text{if } k > 3. \end{cases}$$

Example (Non-linear resonance)

Let $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4\rangle$. Then

$$\mathbf{L}(A): \quad S^3 \xrightarrow{\begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$$

RESONANCE VARIETIES OF PD-ALGEBRAS

- Let A be a PD_m algebra. For $0 \leqslant i \leqslant m$ and $a \in A^1$, the square

$$\begin{array}{ccc} (A^{m-i})^* & \xrightarrow{(\delta_{-a}^{m-i-1})^*} & (A^{m-i-1})^* \\ \mathrm{PD}\uparrow\cong & & \mathrm{PD}\uparrow\cong \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1} \end{array}$$

  commutes up to a sign.

- Consequently,

$$\left(H^i(A,\delta_a)\right)^* \cong H^{m-i}(A,\delta_{-a}).$$

- Hence, for all i and k,

$$\mathcal{R}^i_k(A) = \mathcal{R}^{m-i}_k(A).$$

- In particular, $\mathcal{R}^m_1(A) = \{0\}$.

COROLLARY

Let A be a PD_3 algebra with $b_1(A) = n$. Then

1. $\mathcal{R}^i_0(A) = A^1$.
2. $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$ *and* $\mathcal{R}^2_n(A) = \mathcal{R}^1_n(A) = \{0\}$.
3. $\mathcal{R}^2_k(A) = \mathcal{R}^1_k(A)$ *for* $0 < k < n$.
4. *In all other cases,* $\mathcal{R}^i_k(A) = \emptyset$.

THEOREM

Every PD_3 algebra A decomposes as $A \cong B\#C$, where B are C are PD_3 algebras such that μ_B is irreducible and has the same rank as μ_A, and $\mu_C = 0$. Furthermore, $A^1 \cong B^1 \oplus C^1$ restricts to isomorphisms

$$\mathcal{R}^1_k(A) \cong \mathcal{R}^1_{k-r+1}(B) \times C^1 \cup \mathcal{R}^1_{k-r}(B) \times \{0\} \quad (\forall k \geqslant 0),$$

where $r = \operatorname{corank} \mu_A$. In particular, $\mathcal{R}^1_k(A) = A^1$ for all $k < \operatorname{corank} \mu_A$.

(The *rank* of a form $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.)

- A linear subspace $U \subset V$ is 2-*singular* with respect to a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.
- If $\dim U = 2$, we simply say U is a *singular plane*.
- The *nullity* of μ, denoted $\operatorname{null}(\mu)$, is the maximum dimension of a 2-singular subspace $U \subset V$.
- Clearly, V contains a singular plane if and only if $\operatorname{null}(\mu) \geqslant 2$.
- Let A be a PD_3 algebra. A linear subspace $U \subset A^1$ is 2-singular (with respect to μ_A) if and only if U is isotropic.
- Using a result of A. Sikora [2005], we obtain:

THEOREM

Let A be a PD_3 algebra over an algebraically closed field $\Bbbk$ ($\operatorname{char}(\Bbbk) \neq 2$), and let $\nu = \operatorname{null}(\mu_A)$. If $b_1(A) \geqslant 4$, then

$$\dim \mathcal{R}^1_{\nu-1}(A) \geqslant \nu \geqslant 2.$$

In particular, $\dim \mathcal{R}^1_1(A) \geqslant \nu$.

REAL FORMS AND RESONANCE

- Sikora made the following conjecture: If $\mu\colon \bigwedge^3 V \to \Bbbk$ is a 3-form with $\dim V \geqslant 4$ and if $\operatorname{char}(\Bbbk) \neq 2$, then $\operatorname{null}(\mu) \geqslant 2$.
- Conjecture holds if $n := \dim V$ is even or equal to 5, or if $\Bbbk = \overline{\Bbbk}$.
- Work of J. Draisma and R. Shaw [2010, 2014] implies that the conjecture does not hold for $\Bbbk = \mathbb{R}$ and $n = 7$. We obtain:

THEOREM

Let A be a PD_3 algebra over $\mathbb{R}$. Then $\mathcal{R}^1_1(A) \neq \{0\}$, except when

- $n = 1$, $\mu_A = 0$.
- $n = 3$, $\mu_A = e^1 e^2 e^3$.
- $n = 7$, $\mu_A = -e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^6 + e^2e^4e^5 + e^1e^2e^7 + e^3e^4e^7 + e^5e^6e^7$.

Sketch: If $\mathcal{R}^1_1(A) = \{0\}$, then the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$, and thus a division algebra structure on $\mathbb{R}^{n+1}$, forcing $n = 1, 3$ or 7 by Bott–Milnor/Kervaire [1958].

EXAMPLE

- Let A be the real PD_3 algebra corresponding to octonionic multiplication (defined as above).
- Let A' be the real PD_3 algebra with $\mu_{A'} = e^1e^2e^3 + e^4e^5e^6 + e^1e^4e^7 + e^2e^5e^7 + e^3e^6e^7$.
- Then $\mu_A \sim \mu_{A'}$ over $\mathbb{C}$, and so $A \otimes_{\mathbb{R}} \mathbb{C} \cong A' \otimes_{\mathbb{R}} \mathbb{C}$.
- On the other hand, $A \not\cong A'$ over $\mathbb{R}$, since $\mu_A \not\sim \mu_{A'}$ over $\mathbb{R}$, but also because $\mathcal{R}^1_1(A) = \{0\}$, yet $\mathcal{R}^1_1(A') \neq \{0\}$.
- Both $\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C})$ are projectively smooth conics, and thus are projectively equivalent over $\mathbb{C}$, but

$$\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1^2 + \cdots + x_7^2 = 0\}$$

has only one real point ($x = 0$), whereas

$$\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1x_4 + x_2x_5 + x_3x_6 = x_7^2\}$$

contains the real (isotropic) subspace $\{x_4 = x_5 = x_6 = x_7 = 0\}$.

PFAFFIANS AND RESONANCE

- For a $\Bbbk$-PD$_3$ algebra A, the complex $\mathbf{L}(A) = (A \otimes_{\Bbbk} S, \delta_A)$ looks like

$$A^0 \otimes_{\Bbbk} S \xrightarrow{\delta_A^0} A^1 \otimes_{\Bbbk} S \xrightarrow{\delta_A^1} A^2 \otimes_{\Bbbk} S \xrightarrow{\delta_A^2} A^3 \otimes_{\Bbbk} S \,,$$

  where $\delta_A^0 = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and $\delta_A^2 = (\delta_A^0)^{\top}$, while δ_A^1 is the skew-symmetric matrix whose are entries linear forms in S given by

$$\delta_A^1(e_i) = \sum_{j=1}^{n} \sum_{k=1}^{n} \mu_{jik} e_k^{\vee} \otimes x_j \,.$$

- Recall that $\mathcal{R}_k^1(A) = V(I_{n-k}(\delta_A^1))$. Using work of Buchsbaum and Eisenbud [1977] on Pfaffians of skew-symmetric matrices, we get:

THEOREM

$$\mathcal{R}_{2k}^1(A) = \mathcal{R}_{2k+1}^1(A) = V(\mathrm{Pf}_{n-2k}(\delta_A^1)), \quad \textit{if } n \textit{ is even},$$
$$\mathcal{R}_{2k-1}^1(A) = \mathcal{R}_{2k}^1(A) = V(\mathrm{Pf}_{n-2k+1}(\delta_A^1)), \quad \textit{if } n \textit{ is odd}.$$

- Hence, $A^1 = \mathcal{R}_0^1 = \mathcal{R}_1^1 \supseteq \mathcal{R}_2^1 = \mathcal{R}_3^1 \supseteq \mathcal{R}_4^1 = \cdots$ if $b_1(A)$ is even, and $A^1 = \mathcal{R}_0^1 \supseteq \mathcal{R}_1^1 = \mathcal{R}_2^1 \supseteq \mathcal{R}_3^1 = \mathcal{R}_4^1 \supseteq \cdots$ if $b_1(A)$ is odd.

BOTTOM-DEPTH RESONANCE

THEOREM

Let A be a PD_3 algebra. If μ_A has maximal rank $n \geqslant 3$, then

$$\mathcal{R}^1_{n-2}(A) = \mathcal{R}^1_{n-1}(A) = \mathcal{R}^1_n(A) = \{0\}.$$

Otherwise, write $A = B \# C$, where μ_B is irreducible and $\mu_C = 0$. If $n = \dim A^1$ is at least 3, then $\mathcal{R}^1_{n-2}(A) = \mathcal{R}^1_{n-1}(A) = C^1$.

LEMMA (TURAEV 2002)

Suppose $n \geqslant 3$. There is then a polynomial $\mathrm{Det}(\mu_A) \in S$ such that, if $\delta^1_A(i;j)$ is the sub-matrix obtained from δ^1_A by deleting the i-th row and j-th column, then $\det \delta^1_A(i;j) = (-1)^{i+j} x_i x_j \,\mathrm{Det}(\mu_A)$.

Moreover, if n is even, then $\mathrm{Det}(\mu_A) = 0$, while if n is odd, then $\mathrm{Det}(\mu_A) = \mathrm{Pf}(\mu_A)^2$, where $\mathrm{pf}(\delta^1_A(i;i)) = (-1)^{i+1} x_i \,\mathrm{Pf}(\mu_A)$.

TOP-DEPTH RESONANCE

Suppose $\dim_{\Bbbk} V = 2g+1 > 1$. We say that a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ is *generic* (in the sense of Berceanu–Papadima [1994]) if there is a $v \in V$ such that the 2-form $\gamma_v \in V^* \wedge V^*$ given by $\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v)$ for $a, b \in V$ has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^*$.

THEOREM

Let A be a PD_3 algebra. Then

$$\mathcal{R}_1^1(A) = \begin{cases} \emptyset & \text{if } n = 0; \\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu \text{ has rank } 3; \\ V(\mathrm{Pf}(\mu_A)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_A \text{ is BP-generic}; \\ A^1 & \text{otherwise.} \end{cases}$$

EXAMPLE

Let $M = \Sigma_g \times S^1$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^{g} a_i b_i c$ is BP-generic, and $\mathrm{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1^1(M) = \{x_{2g+1} = 0\}$. In fact,

$$\mathcal{R}_1^1 = \cdots = \mathcal{R}_{2g-2}^1 \text{ and } \mathcal{R}_{2g-1}^1 = \mathcal{R}_{2g}^1 = \mathcal{R}_{2g+1}^1 = \{0\}.$$

As a corollary, we recover a closely related result, proved by Draisma and Shaw [2010] by very different methods.

COROLLARY

Let V be a $\mathbb{k}$-vector space of odd dimension $n \geqslant 5$ and let $\mu \in \bigwedge^3 V^*$. Then the union of all singular planes is either all of V or a hypersurface defined by a homogeneous polynomial in $\mathbb{k}[V]$ of degree $(n-3)/2$.

For $\mu \in \bigwedge^3 V^*$, there is another genericity condition, due to P. De Poi, D. Faenzi, E. Mezzetti, and K. Ranestad [2017]: $\operatorname{rank}(\gamma_v) > 2$, for all non-zero $v \in V$. We may interpret some of their results, as follows.

THEOREM (DFMR)

Let A be a PD_3 algebra over $\mathbb{C}$, and suppose μ_A is generic. Then:

- *If n is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leqslant 7$, and singular in codimension 5 if $n \geqslant 9$.*
- *If n is even, then $\mathcal{R}^1_2(A)$ has codim 3 and degree $\frac{1}{4}\binom{n-2}{3}+1$; it is smooth if $n \leqslant 10$, and singular in codimension 7 if $n \geqslant 12$.*

RESONANCE VARIETIES OF 3-FORMS OF LOW RANK

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3$
5	125+345	$\{x_5 = 0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4$
6	123+456	$\mathbb{C}^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3 = x_5 = x_6 = 0\}$	0

n	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3 = \mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367	$\{x_7 = 0\}$	$\{x_7 = 0\}$	0
	456+147+257+367	$\{x_7 = 0\}$	$\{x_4 = x_5 = x_6 = x_7 = 0\}$	0
	123+456+147	$\{x_1 = 0\} \cup \{x_4 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_4 = x_5 = x_6 = 0\}$	0
	123+456+147+257	$\{x_1 x_4 + x_2 x_5 = 0\}$	$\{x_1 = x_2 = x_4 = x_5 = x_7^2 - x_3 x_6 = 0\}$	0
	123+456+147+257+367	$\{x_1 x_4 + x_2 x_5 + x_3 x_6 = x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4 = \mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7 = 0\}$	$\{x_3 = x_5 = x_7 = x_8 = 0\} \cup \{x_1 = x_3 = x_4 = x_5 = x_7 = 0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5 = x_7 = 0\}$	$\{x_3 = x_4 = x_5 = x_7 = x_1 x_8 + x_6^2 = 0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = 0\}$	$\{x_1 = x_3 = x_4 = x_5 = x_2 x_6 + x_7 x_8 = 0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1 = x_5 = 0\} \cup \{x_3 = x_4 = x_5 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3 = x_5 = x_1 x_4 - x_7^2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$
	147+268+358	$\mathbb{C}^8$	$\{x_1 = x_4 = x_7 = 0\} \cup \{x_8 = 0\}$	$\{x_1 = x_4 = x_7 = x_8 = 0\} \cup \{x_2 = x_3 = x_5 = x_6 = x_8 = 0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3$	$L_1 \cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	$L_1 \cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1 \cup L_2 \cup L_3 \cup L_4$	$L'_1 \cup L'_2 \cup L'_3$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1 \cup C_2 \cup C_3$	$L_1 \cup L_2 \cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1 \cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1 = \cdots = f_{20} = 0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1 = \cdots = g_{20} = 0\}$	0