FUNDAMENTAL GROUPS IN ALGEBRAIC GEOMETRY AND THREE-DIMENSIONAL TOPOLOGY

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FUNDAMENTAL GROUPS OF MANIFOLDS

- Every finitely presented group π can be realized as $\pi = \pi_1(M)$, for some smooth, compact, connected manifold M^n of dim $n \geqslant 4$.
- M^n can be chosen to be orientable.
- If n even, $n \geqslant 4$, then M^n can be chosen to be symplectic (Gompf).
- If n even, $n \geqslant 6$, then M^n can be chosen to be complex (Taubes).
- Requiring that $n = 3$ puts severe restrictions on the (closed) 3-manifold group $\pi = \pi_1(M^3)$.

KÄHLER GROUPS

- A *Kähler manifold* is a compact, connected, complex manifold, with a Hermitian metric h such that $\omega = \text{im}(h)$ is a closed 2-form.
- Smooth, complex projective varieties are Kähler manifolds.
- A group π is called a *Kähler group* if $\pi = \pi_1(M)$, for some Kähler manifold M.
- The group π is a *projective group* if M can be chosen to be a projective manifold.
- The classes of Kähler and projective groups are closed under finite direct products and passing to finite-index subgroups.
- Every finite group is a projective group. [Serre $\sim$1955]

- The Kähler condition puts strong restrictions on π, e.g.:
  - π is finitely presented.
  - $b_1(\pi)$ is even. [by Hodge theory]
  - π is 1-formal [Deligne–Griffiths–Morgan–Sullivan 1975]
    (i.e., its Malcev Lie algebra $\mathfrak{m}(\pi) := \operatorname{Prim}(\widehat{\mathbb{Q}[\pi]})$ is quadratic)
  - π cannot split non-trivially as a free product. [Gromov 1989]
- Problem: Are all Kähler groups projective groups?
- Problem [Serre]: Characterize the class of projective groups.

QUASI-PROJECTIVE GROUPS

- A group π is said to be a *quasi-projective group* if $\pi = \pi_1(M \backslash D)$, where M is a smooth, projective variety and D is a divisor.
- Qp groups are finitely presented. The class of qp groups is closed under direct products and passing to finite-index subgroups.
- For a qp group π,
  - $b_1(\pi)$ can be arbitrary (e.g., the free groups F_n).
  - π may be non-1-formal (e.g., the Heisenberg group).
  - π can split as a non-trivial free product (e.g., $F_2 = \mathbb{Z} * \mathbb{Z}$).

COMPLEMENTS OF HYPERSURFACES

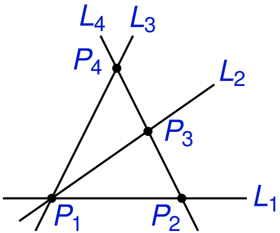
- A subclass of quasi-projective groups consists of fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$,
$$\pi = \pi_1(\mathbb{CP}^n \setminus \{f = 0\}), \quad f \in \mathbb{C}[z_0, \dots, z_n] \text{ homogeneous.}$$
- All such groups are 1-formal. [Kohno 1983]
- By the Lefschetz hyperplane sections theorem, $\pi = \pi_1(\mathbb{CP}^2 \setminus \mathcal{C})$, for some plane algebraic curve $\mathcal{C}$.
- Zariski asked Van Kampen to find presentations for such groups.
- Using the Alexander polynomial, Zariski showed that π is *not* determined by the combinatorics of $\mathcal{C}$, but depends on the position of its singularities.

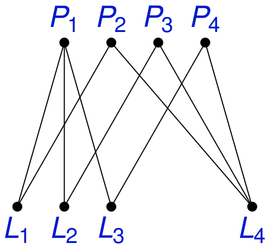
PROBLEM (ZARISKI)

Is $\pi = \pi_1(\mathbb{CP}^2 \setminus \mathcal{C})$ residually finite, *i.e., is the map to the profinite completion,* $\pi \to \pi^{\mathrm{alg}} := \varprojlim_{G \triangleleft_{\text{f.i.}} \pi} \pi/G$, *injective?*

HYPERPLANE ARRANGEMENTS

- Even more special are the *arrangement groups*, i.e., the fundamental groups of complements of complex hyperplane arrangements (or, equivalently, complex line arrangements).
- Let $\mathcal{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial $f = \prod_{L\in\mathcal{A}} f_L$, with f_L linear forms so that $L = \mathbb{P}(\ker(f_L))$.
- The combinatorics of $\mathcal{A}$ is encoded in the *intersection poset*, $\mathcal{L}(\mathcal{A})$, with $\mathcal{L}_1(\mathcal{A}) = \{\text{lines}\}$ and $\mathcal{L}_2(\mathcal{A}) = \{\text{intersection points}\}$.





- Let $U(\mathcal{A}) = \mathbb{CP}^2 \setminus \bigcup_{L \in \mathcal{A}} L$. The group $\pi = \pi_1(U(\mathcal{A}))$ has a finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$, and $\prod x_i = 1$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where $\alpha_1, \dots \alpha_s \in P_n \subset \operatorname{Aut}(F_n)$, and $s = |\mathcal{L}_2(\mathcal{A})|$.
- Let $\gamma_1(\pi) = \pi, \ \ \gamma_2(\pi) = \pi' = [\pi, \pi], \ \ \gamma_k(\pi) = [\gamma_{k-1}(\pi), \pi]$, be the lower central series of π. Then:
  - $\pi_{\text{ab}} = \pi / \gamma_2$ equals $\mathbb{Z}^{n-1}$.
  - π / γ_3 is determined by $L(\mathcal{A})$.
  - π / γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$ (G. Rybnikov).

PROBLEM (ORLIK)

Is π torsion-free?

- Answer is yes if $U(\mathcal{A})$ is a $K(\pi, 1)$. This happens if the cone on $\mathcal{A}$ is a simplicial arrangement (Deligne), or supersolvable (Terao).

ARTIN GROUPS

- Let $\Gamma = (V, E)$ be a finite, simple graph, and let $\ell \colon E \to \mathbb{Z}_{\geqslant 2}$ be an edge-labeling. The associated *Artin group*:

$$A_{\Gamma,\ell} = \langle v \in V \mid \underbrace{vwv\cdots}_{\ell(e)} = \underbrace{wvw\cdots}_{\ell(e)}, \text{ for } e = \{v, w\} \in E \rangle.$$

- If (Γ, ℓ) is Dynkin diagram of type A_{n-1} with $\ell(\{i, i+1\}) = 3$ and $\ell(\{i, j\}) = 2$ otherwise, then $A_{\Gamma,\ell}$ is the braid group B_n.
- If $\ell(e) = 2$, for all $e \in E$, then

$$A_{\Gamma} = \langle v \in \mathrm{V} \mid vw = wv \text{ if } \{v, w\} \in \mathrm{E} \rangle.$$

  is the *right-angled Artin group* associated to Γ.
- $\Gamma \cong \Gamma' \Leftrightarrow A_{\Gamma} \cong A_{\Gamma'}$

  [Kim–Makar-Limanov–Neggers–Roush 80 / Droms 87]

The corresponding *Coxeter group*,

$$W_{\Gamma,\ell} = A_{\Gamma,\ell}/\langle v^2 = 1 \mid v \in V\rangle,$$

fits into exact sequence $1 \to P_{\Gamma,\ell} \to A_{\Gamma,\ell} \to W_{\Gamma,\ell} \to 1$.

THEOREM (BRIESKORN 1971)

If $W_{\Gamma,\ell}$ is finite, then $G_{\Gamma,\ell}$ is quasi-projective.

Idea: let

- $\mathcal{A}_{\Gamma,\ell} =$ reflection arrangement of type $W_{\Gamma,\ell}$ (over $\mathbb{C}$)
- $X_{\Gamma,\ell} = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_{\Gamma,\ell}} H$, where $n = |\mathcal{A}_{\Gamma,\ell}|$
- $P_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell})$

then:

$$A_{\Gamma,\ell} = \pi_1(X_{\Gamma,\ell}/W_{\Gamma,\ell}) = \pi_1(\mathbb{C}^n \setminus \{\delta_{\Gamma,\ell} = 0\})$$

THEOREM (KAPOVICH–MILLSON 1998)

There exist infinitely many (Γ, ℓ) such that $A_{\Gamma,\ell}$ is not *quasi-projective.*

KÄHLER GROUPS VS OTHER GROUPS

QUESTION (DONALDSON–GOLDMAN 1989)

Which 3-manifold groups are Kähler groups?

THEOREM (DIMCA–S. 2009)

Let π be the fundamental group of a closed 3-manifold. Then π is a Kähler group $\Longleftrightarrow$ π is a finite subgroup of $O(4)$, acting freely on S^3.

Alternative proofs: Kotschick (2012), Biswas, Mj, and Seshadri (2012).

THEOREM (FRIEDL–S. 2014)

Let N be a 3-manifold with non-empty, toroidal boundary. If $\pi_1(N)$ is a Kähler group, then $N \cong S^1 \times S^1 \times I$.

Generalization by Kotschick: If $\pi_1(N)$ is an infinite Kähler group, then $\pi_1(N)$ is a surface group.

THEOREM (DIMCA–PAPADIMA–S. 2009)

Let Γ be a finite simple graph, and A_Γ the corresponding RAAG. The following are equivalent:

1. *A_Γ is a Kähler group.*
2. *A_Γ is a free abelian group of even rank.*
3. *Γ is a complete graph on an even number of vertices.*

THEOREM (S. 2011)

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, with group $\pi = \pi_1(U(\mathcal{A}))$. The following are equivalent:

1. *π is a Kähler group.*
2. *π is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an odd number of lines in general position.*

QUASI-PROJECTIVE GROUPS VS OTHER GROUPS

THEOREM (DIMCA–PAPADIMA–S. 2011)

Let π be the fundamental group of a closed, orientable 3-manifold. Assume π is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-projective manifold* X.
2. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(N))$, *where* N *is either* S^3, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

THEOREM (FRIEDL–S. 2014)

Let N be a 3-mfd with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all prime components of N are graph manifolds.

In particular, the fundamental group of a hyperbolic 3-manifold with empty or toroidal boundary is never a qp-group.

THEOREM (DPS 2009)

A right-angled Artin group A_Γ is a quasi-projective group if and only if Γ is a complete multipartite graph $K_{n_1,\dots,n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$, in which case $A_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.

THEOREM (S. 2011)

Let $\pi = \pi_1(U(\mathcal{A}))$ be an arrangement group. The following are equivalent:

1. *π is a RAAG.*
2. *π is a finite direct product of finitely generated free groups.*
3. *$\mathcal{G}(\mathcal{A})$ is a forest.*

Here $\mathcal{G}(\mathcal{A})$ is the 'multiplicity' graph, with

- vertices: points $P \in \mathcal{L}_2(\mathcal{A})$ with multiplicity at least 3;
- edges: $\{P, Q\}$ if $P, Q \in L$, for some $L \in \mathcal{A}$.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.
- Let $\text{Char}(X) = \text{Hom}(\pi, \mathbb{C}^*)$ be the affine algebraic group of $\mathbb{C}$-valued, multiplicative characters on π.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}^q_s(X) = \{\rho \in \text{Char}(X) \mid \dim_{\mathbb{C}} H_q(X, \mathbb{C}_\rho) \geqslant s\}.$$

  Here, $\mathbb{C}_\rho$ is the local system defined by ρ, i.e, $\mathbb{C}$ viewed as a $\mathbb{C}\pi$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{C}_\rho) = H_i(C_\bullet(\widetilde{X}, \Bbbk) \otimes_{\mathbb{C}\pi} \mathbb{C}_\rho)$.
- These loci are Zariski closed subsets of the character group.
- The sets $\mathcal{V}^1_s(X)$ depend only on π/π''.

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{C}\mathbb{Z} = \mathbb{C}[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}, \mathbb{C}) : \ 0 \longrightarrow \mathbb{C}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{C}[t^{\pm 1}] \longrightarrow 0 .$$

For $\rho \in \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) = \mathbb{C}^*$, we get

$$C_*(\widetilde{S^1}, \mathbb{C}) \otimes_{\mathbb{C}\mathbb{Z}} \mathbb{C}_\rho : \ 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0 ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence: $\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$ and $\mathcal{V}^i_s(S^1) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C} \backslash \{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\mathbb{C}^*)^n$. Then:

$$\mathcal{V}^1_s(\mathbb{C} \backslash \{n \text{ points}\}) = \begin{cases} (\mathbb{C}^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

Resonance varieties

- Let $A = H^*(X, \mathbb{C})$. Then: $a \in A^1 \Rightarrow a^2 = 0$.
- We thus get a cochain complex

$$(A, \cdot a)\colon\; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \longrightarrow \cdots .$$

- The *resonance varieties* of X are the jump loci for the cohomology of this complex

$$\mathcal{R}^q_s(X) = \{a \in A^1 \mid \dim_{\Bbbk} H^q(A, \cdot a) \geqslant s\}$$

- E.g., $\mathcal{R}^1_1(X) = \{a \in A^1 \mid \exists b \in A^1,\ b \neq \lambda a,\ ab = 0\}$.
- These loci are *homogeneous* subvarieties of $A^1 = H^1(X, \mathbb{C})$.

Example

- $\mathcal{R}^1_1(T^n) = \{0\}$, for all $n > 0$.
- $\mathcal{R}^1_1(\mathbb{C}\backslash\{n \text{ points}\}) = \mathbb{C}^n$, for all $n > 1$.

THE TANGENT CONE THEOREM

- Given a subvariety $W \subset (\mathbb{C}^*)^n$, let $\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W, \ \forall \lambda \in \mathbb{C}\}$.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.
- (Libgober 2002/DPS 2009)
$$\tau_1(\mathcal{V}^i_s(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$
- (DPS 2009/DP 2014): Suppose X is a k-formal space. Then, for each $i \leqslant k$ and $s > 0$,
$$\tau_1(\mathcal{V}^i_s(X)) = \mathrm{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X).$$
- Consequently, $\mathcal{R}^i_s(X)$ is a union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.

QUASI-PROJECTIVE VARIETIES

THEOREM (ARAPURA 1997, ..., BUDUR–WANG 2015)

Let X be a smooth, quasi-projective variety. Then each $\mathcal{V}^i_s(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$.

In particular, if π is a quasi-projective group, then all components of $V^1_1(\pi)$ are torsion-translated subtori of $\mathrm{Hom}(\pi, \mathbb{C}^*)$.

The *Alexander polynomial* of a f.p. group π is the Laurent polynomial Δ_π in $\Lambda := \mathbb{C}[\pi_{\mathrm{ab}}/\mathrm{Tors}]$ gotten by taking the gcd of the maximal minors of a presentation matrix for the Λ-module $H_1(\pi, \Lambda)$.

THEOREM (DIMCA–PAPADIMA–S. 2008)

Let π be a quasi-projective group.

- *If $b_1(\pi) \neq 2$, then the Newton polytope of Δ_π is a line segment.*
- *If π is a Kähler group, then $\Delta_\pi \doteq \mathrm{const}$.*

Toric complexes and RAAGs

- Given a simplicial complex L on n vertices, let T_L be the corresponding subcomplex of the n-torus T^n.
- Then $\pi_1(T_L) = A_\Gamma$, where $\Gamma = L^{(1)}$ and $K(A_\Gamma, 1) = T_{\Delta_\Gamma}$.
- Identify $H^1(T_L, \mathbb{C})$ with $\mathbb{C}^{\mathsf{V}} = \operatorname{span}\{v \mid v \in \mathsf{V}\}$.

Theorem (Papadima–S. 2010)

$$\mathcal{R}^i_s(T_L) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{L_{\mathsf{W}}}(\sigma), \mathbb{C}) \geqslant s}} \mathbb{C}^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\operatorname{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular: $$\mathcal{R}^1_1(A_\Gamma) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \mathbb{C}^{\mathsf{W}}.$$

CLOSED 3-MANIFOLDS

- Let M be a closed, orientable 3-manifold.
- An orientation class $[M] \in H_3(M, \mathbb{Z}) \cong \mathbb{Z}$ defines an alternating 3-form μ_M on $H^1(M, \mathbb{Z})$ by $\mu_M(a, b, c) = \langle a \cup b \cup c, [M] \rangle$.

THEOREM (S. 2016)

Set $n = b_1(M)$. Then $\mathcal{R}^1_1(M) = \emptyset$ if $n = 0$, $\mathcal{R}^1_1(M) = \{0\}$ if $n = 1$, and otherwise

$$\mathcal{R}^1_1(M) = V(\mathrm{Det}(\mu_M)) = \begin{cases} H^1(M, \mathbb{C}) & \text{if } n \text{ is even,} \\ V(\mathrm{Pf}(\mu_M)) & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM (S. 2016)

If $b_1(M) \neq 2$, then $\mathrm{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M)$.

In general, $\tau_1(\mathcal{V}^1_1(M)) \subsetneq \mathrm{TC}_1(\mathcal{V}^1_1(M))$, in which case M is not 1-formal.

THE RFRp PROPERTY

Joint work with Thomas Koberda (arxiv:1604.02010)

Let G be a finitely generated group and let p be a prime.

We say that G is *residually finite rationally* p if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that

1. $G_{i+1} \lhd G_i$.
2. $\bigcap_{i\geqslant 0} G_i = \{1\}$.
3. G_i / G_{i+1} is an elementary abelian p-group.
4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.

Remarks:

- We may assume that each $G_i \lhd G$.
- G is RFRp if and only if $\operatorname{rad}_p(G) := \bigcap_i G_i$ is trivial.
- For each prime p, there exists a finitely presented group G_p which is RFRp, but not RFRq for any prime $q \neq p$.

- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite and residually nilpotent.
- G RFRp $\Rightarrow$ G torsion-free.
- G finitely presented and RFRp $\Rightarrow$ G has solvable word problem.
- The class of RFRp groups is closed under taking subgroups, finite direct products, and finite free products.
- Finitely generated free groups F_n, surface groups $\pi_1(\Sigma_g)$, and right-angled Artin groups A_Γ are RFRp, for all p.
- Finite groups and non-abelian nilpotent groups are *not* RFRp, for any p.

THEOREM (A TITS ALTERNATIVE FOR RFRp GROUPS)

If G is a finitely presented group which is RFRp for infinitely many primes p, then either G is abelian or G is large (i.e., it virtually surjects onto a non-abelian free group).

A COMBINATION THEOREM

- The *RFRp topology* on a group G has basis the cosets of the standard RFRp filtration $\{G_i\}$ of G.
- G is RFRp iff this topology is Hausdorff.

THEOREM

Fix a prime p. Let $X = X_\Gamma$ be a finite graph of connected, finite CW-complexes with vertex spaces $\{X_v\}_{v\in V(\Gamma)}$ and edge spaces $\{X_e\}_{e\in E(\Gamma)}$ satisfying the following conditions:

1. *For each $v \in V(\Gamma)$, the group $\pi_1(X_v)$ is RFRp.*
2. *For each $v \in V(\Gamma)$, the RFRp topology on $\pi_1(X)$ induces the RFRp topology on $\pi_1(X_v)$ by restriction.*
3. *For each $e \in E(\Gamma)$ and each $v \in e$, the subgroup $\phi_{e,v}(\pi_1(X_e))$ of $\pi_1(X_v)$ is closed in the RFRp topology on $\pi_1(X_v)$.*

Then $\pi_1(X)$ is RFRp.

BOUNDARY MANIFOLDS OF LINE ARRANGEMENTS

- Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, and let N be a regular neighborhood of $\bigcup_{L\in\mathcal{A}} L$.
- The *boundary manifold* of $\mathcal{A}$ is $M = \partial N$, a compact, orientable, smooth manifold of dimension 3.

EXAMPLE

Let $\mathcal{A}$ be a pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1^n - z_2^n$. If $n = 1$, then $M = S^3$. If $n > 1$, then $M = \sharp^{n-1} S^1 \times S^2$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1(z_2^{n-1} - z_3^{n-1})$. Then $M = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- M is a graph-manifold M_Γ, where Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L, P) \mid P \in L\}$.
- For each $v \in V(\Gamma)$, there is a vertex manifold $M_v = S^1 \times S_v$, with $S_v = S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w}$.
- Vertex manifolds are glued along edge manifolds $M_e = S^1 \times S^1$ via flips.
- The boundary manifold of a line arrangement in $\mathbb{C}^2$ is defined as $M = \partial N \cap D^4$, for some sufficiently large 4-ball D^4.

THEOREM

If M is the boundary manifold of a line arrangement in $\mathbb{C}^2$, then $\pi_1(M)$ is RFRp, for all primes p.

CONJECTURE

Arrangement groups are RFRp, for all primes p.