Polyhedral products, duality properties, and cohomology jump loci

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Discrete Geometry and Topological Combinatorics Seminar

Freie University Berlin

November 25, 2022

Polyhedral products

- Let (X, A) be a pair of topological spaces, and let L be a simplicial complex on vertex set $[m]$.
- The corresponding *polyhedral product* (or, *generalized moment-angle complex*) is defined as

$$\mathcal{Z}_L(X, A) = \bigcup_{\sigma \in L} (X, A)^\sigma \subset X^{\times m},$$

where $(X, A)^\sigma = \{x \in X^{\times m} \mid x_i \in A \text{ if } i \notin \sigma\}$.

- Homotopy invariance:

$$(X, A) \simeq (X', A') \implies \mathcal{Z}_L(X, A) \simeq \mathcal{Z}_L(X', A').$$

- Converts simplicial joins to direct products:

$$\mathcal{Z}_{K*L}(X, A) \cong \mathcal{Z}_K(X, A) \times \mathcal{Z}_L(X, A).$$

- Takes a cellular pair (X, A) to a cellular subcomplex of $X^{\times m}$.

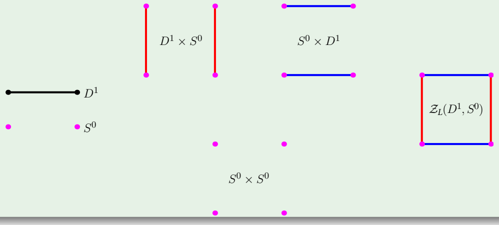
The usual moment-angle complexes are:

- Complex moment-angle complex, $\mathcal{Z}_L(D^2, S^1)$.
  - $\pi_1 = \pi_2 = \{1\}$.
- Real moment-angle complex, $\mathcal{Z}_L(D^1, S^0)$.
  - $\pi_1 = W_L'$, the derived subgroup of W_Γ, the right-angled Coxeter group associated to $\Gamma = L^{(1)}$.

EXAMPLE

Let $L =$ two points. Then:

$$\mathcal{Z}_L(D^2, S^1) = D^2 \times S^1 \cup S^1 \times D^2 = S^3$$
$$\mathcal{Z}_L(D^1, S^0) = D^1 \times S^0 \cup S^0 \times D^1 = S^1$$

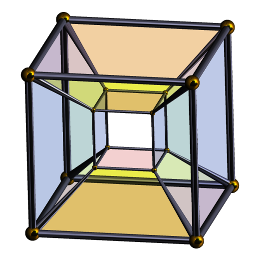


Example

Let L be a circuit on 4 vertices. Then:

$$\mathcal{Z}_L(D^2, S^1) = S^3 \times S^3$$

$$\mathcal{Z}_L(D^1, S^0) = S^1 \times S^1$$



Example

More generally, let L be an m-gon. Then:

$$\mathcal{Z}_L(D^2, S^1) = \#_{r=1}^{m-3}\, r \cdot \binom{m-2}{r+1} S^{r+2} \times S^{m-r}.$$

(McGavran 1979)

$$\mathcal{Z}_L(D^1, S^0) = \text{an orientable surface of genus } 1 + 2^{m-3}(m-4).$$

(Coxeter 1937)

- If $(M, \partial M)$ is a compact manifold of dimension d, and L is a PL-triangulation of S^m on n vertices, then $\mathcal{Z}_L(M, \partial M)$ is a compact manifold of dimension $(d-1)n + m + 1$.

- (Bosio–Meersseman 2006) If K is a *polytopal* triangulation of S^m, then
  - $\mathcal{Z}_L(D^2, S^1)$ if $n + m + 1$ is even, or
  - $\mathcal{Z}_L(D^2, S^1) \times S^1$ if $n + m + 1$ is odd

  is a complex manifold.

- This construction generalizes the classical constructions of complex structures on $S^{2p-1} \times S^1$ (Hopf) and $S^{2p-1} \times S^{2q-1}$ (Calabi–Eckmann).

- In general, the resulting complex manifolds are *not* symplectic, thus, not Kähler. In fact, they may even be non-formal (Denham–Suciu 2007, Grbić–Linton 2021).

- The GMAC construction enjoys nice functoriality properties in both arguments. E.g:
  - Let $f\colon (X,A)\to (Y,B)$ be a (cellular) map. Then $f^{\times n}\colon X^{\times n}\to Y^{\times n}$ restricts to a (cellular) map $\mathcal{Z}_L(f)\colon \mathcal{Z}_L(X,A)\to \mathcal{Z}_L(Y,B)$.
- Much is known from work of M. Davis about the fundamental group and the asphericity problem for $\mathcal{Z}_L(X)=\mathcal{Z}_L(X,*)$. E.g.:
  - $\pi_1(\mathcal{Z}_L(X,*))$ is the graph product of $G_v=\pi_1(X,*)$ along the graph $\Gamma=L^{(1)}=(\mathsf{V},\mathsf{E})$, where

$$\mathrm{Prod}_\Gamma(G_v)=\mathop{\ast}_{v\in \mathsf{V}} G_v/\{[g_v,g_w]=1 \text{ if } \{v,w\}\in \mathsf{E},\, g_v\in G_v,\, g_w\in G_w\}.$$

  - Suppose X is aspherical. Then: $\mathcal{Z}_L(X,*)$ is aspherical iff L is a flag complex.

Toric complexes

- Let L be a simplicial complex on vertex set $\mathsf{V} = \{v_1, \dots, v_m\}$.
- Let $T_L = \mathcal{Z}_L(S^1, *)$ be the subcomplex of T^m obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a connected, minimal CW-complex, of dimension $\dim L + 1$.
- T_L is formal (Notbohm–Ray 2005).
- (Kim–Roush 1980, Charney–Davis 1995) The cohomology algebra $H^*(T_L, \Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle L\rangle = \bigwedge V^*/(v_\sigma^* \mid \sigma \notin L),$$

where $\Bbbk = \mathbb{Z}$ or a field, V is the free $\Bbbk$-module on V, and $V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$, while $v_\sigma^* = v_{i_1}^* \cdots v_{i_s}^*$ for $\sigma = \{i_1, \dots, i_s\}$.

Right angled Artin groups

- The fundamental group $\Gamma_\Gamma := \pi_1(T_L, *)$ is the RAAG associated to the graph $\Gamma := L^{(1)} = (\mathsf{V}, \mathsf{E})$,
$$G_\Gamma = \langle v \in V \mid [v,w] = 1 \text{ if } \{v,w\} \in E \rangle.$$
- Moreover, $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.
- (Kim–Makar-Limanov–Neggers–Roush 1980, Droms 1987)
$$\Gamma \cong \Gamma' \iff G_\Gamma \cong G_{\Gamma'}.$$
- (Papadima–S. 2006) The associated graded Lie algebra of G_Γ has (quadratic) presentation
$$\operatorname{gr}(G_\Gamma) = \mathbb{L}(\mathsf{V})/([v,w] = 0 \text{ if } \{v,w\} \in \mathsf{E}).$$
- (Duchamp–Krob 1992, PS06) The lower central series quotients of G_Γ are torsion-free, with ranks ϕ_k given by
$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t),$$
where $P_\Gamma(t) = \sum_{k \geqslant 0} f_k(\Delta_\Gamma) t^k$ is the clique polynomial of Γ.

Chen ranks

- The *Chen Lie algebra* of a f.g. group π is the associated graded Lie algebra of its maximal metabelian quotient, $\operatorname{gr}(\pi/\pi'')$.
- Write $\theta_k(\pi) = \operatorname{rank} \operatorname{gr}_k(\pi/\pi'')$ for the Chen ranks.
- (K.-T. Chen 1951) $\operatorname{gr}(F_n/F_n'')$ is torsion-free, with ranks $\theta_1 = n$ and $\theta_k = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.
- (PS 06) $\operatorname{gr}(G_\Gamma/G_\Gamma'')$ is torsion-free, with ranks given by $\theta_1 = |\mathsf{V}|$ and

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right).$$

- Here $Q_\Gamma(t) = \sum_{j\geqslant 2} c_j(\Gamma) t^j$ is the “cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W}\subset\mathsf{V}\colon\, |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

Example

Let Γ be a pentagon, and Γ' a square with an edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 + 5t + 5t^2$, and so

$$\phi_k(G_\Gamma) = \phi_k(G_{\Gamma'}), \quad \text{for all } k \geqslant 1.$$

- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so

$$\theta_k(G_\Gamma) \neq \theta_k(G_{\Gamma'}), \quad \text{for } k \geqslant 4.$$

Cohomology jump loci

- Let X be a connected, finite CW-complex X with $\pi := \pi_1(X)$.
- Fix a field $\Bbbk$ and set $A = H^{\bullet}(X, \Bbbk)$. If $\operatorname{char}(\Bbbk) = 2$, assume $H_1(X, \mathbb{Z})$ is torsion-free. Then, for each $a \in A^1$, we have $a^2 = 0$, and so we get a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2 \longrightarrow \cdots$.
- The *resonance varieties* of X are defined as

$$\mathcal{R}^i_s(X) = \{a \in A^1 \mid \dim H^i(A, \cdot a) \geqslant s\}.$$

- They are Zariski closed, homogeneous subsets of $A^1 = H^1(X, \Bbbk)$.
- The *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems,

$$\mathcal{V}^i_s(X, \Bbbk) = \{\rho \in \operatorname{Hom}(\pi, \Bbbk^*) \mid \dim H_i(X, \Bbbk_\rho) \geqslant s\}.$$

- These loci are Zariski closed subsets of the character group. For $i = 1$, they depend only on π/π'' (and $\Bbbk$).

Jump loci of toric complexes

For a field $\mathbb{k}$, identify $H^1(T_L, \mathbb{k}) = \mathbb{k}^{\mathsf{V}}$, the $\mathbb{k}$-vector space with basis V.

Theorem (Papadima–S. 2009)

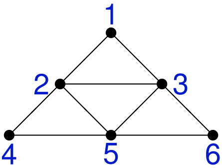
$$\mathcal{R}^i_s(T_L, \mathbb{k}) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{k}} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{k})\geqslant s}} \mathbb{k}^{\mathsf{W}},$$

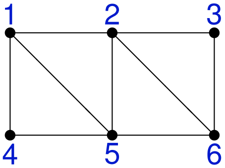
where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular,

$$\mathcal{R}^1_1(G_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \mathbb{k}^{\mathsf{W}}.$$

Similar formulas hold for the characteristic varieties $\mathcal{V}^i_s(T_L, \mathbb{k})$.





EXAMPLE

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, G_Γ and $G_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}_1(G_\Gamma, \Bbbk) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}_1(G_{\Gamma'}, \Bbbk) = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

PROPAGATION OF JUMP LOCI

- We say that the resonance varieties of a graded algebra $A = \bigoplus_{i=0}^{n} A^i$ *propagate* if

$$\mathcal{R}_1^1(A) \subseteq \cdots \subseteq \mathcal{R}_1^n(A).$$

- (Eisenbud–Popescu–Yuzvinsky 2003) If $M(\mathcal{A})$ is the complement of a hyperplane arrangement, then the resonance varieties of the Orlik–Solomon algebra $A = H^*(M(\mathcal{A}), \mathbb{C})$ propagate.

- The resonance varieties of $A = H^*(T_L, \Bbbk)$ may not propagate. E.g., if $L =$  , then $\mathcal{R}_1^1(A) = \Bbbk^4$, yet $\mathcal{R}_1^2(A) = \Bbbk^2 \cup \Bbbk^2$.

THEOREM (DENHAM–S.–YUZVINSKY 2016/17)

Suppose the $\Bbbk$-dual of A has a linear free resolution over $E = \bigwedge A^1$. Then the resonance varieties of A propagate.

Duality spaces

In order to study propagation of jump loci in a topological setting, we turn to a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

Abelian duality spaces

We introduce in (DSY18) an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathsf{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathsf{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathsf{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathsf{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathsf{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathsf{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- The two notions of duality are independent.

Theorem (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \Bbbk^*$ satisfies $H^i(X, \Bbbk_\rho) \neq 0$, then $H^j(X, \Bbbk_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

Corollary (DSY)

Let X be an abelian duality space of dimension n. Then:

- *The characteristic varieties propagate: $\mathcal{V}^1_1(X,\Bbbk) \subseteq \cdots \subseteq \mathcal{V}^n_1(X,\Bbbk)$.*
- $\dim_{\Bbbk} H^1(X,\Bbbk) \geqslant n-1$.
- *If $n \geqslant 2$, then $H^i(X,\Bbbk) \neq 0$, for all $0 \leqslant i \leqslant n$.*

Proposition (DSY)

Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even and non-zero, then the resonance varieties of M do not propagate.

Example

- Let M be the 3-dimensional Heisenberg nilmanifold.
- Characteristic varieties propagate: $\mathcal{V}^i_1(M,\Bbbk) = \{1\}$ for $i \leqslant 3$.
- Resonance does not propagate: $\mathcal{R}^1_1(M,\Bbbk) = \Bbbk^2$, $\mathcal{R}^3_1(M,\Bbbk) = 0$.

Arrangements of smooth hypersurfaces

Theorem (Denham–S. 2018)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

- *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
- *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then:

- *U is both a duality space and an abelian duality space of dimension n.*
- *If A is a finite-dimensional representation of $\pi = \pi_1(U)$, and if $A^{\gamma_g} = 0$ for all g in a building set $\mathcal{G}_X$, for some $X \in L(\mathcal{A})$, then $H^i(U, A) = 0$ for all $i \neq n$.*
- *The ℓ_2-Betti numbers of U vanish for all $i \neq n$.*

Linear, elliptic, and toric arrangements

Theorem (DS17)

Suppose that $\mathcal{A}$ is one of the following:

- *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
- *A non-empty elliptic arrangement in E^n;*
- *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n - r$, $n + r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

- Davis, Januszkiewicz, Leary, and Okun (2011);
- Levin and Varchenko (2012);
- Davis and Settepanella (2013), Esterov and Takeuchi (2014).

The Cohen–Macaulay property

- A simplicial complex L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology $\widetilde{H}^{\bullet}(\operatorname{lk}(\sigma), \mathbb{Z})$ is concentrated in degree $\dim L - |\sigma|$ and is torsion-free.
- An analogous definition works over any coefficient field $\mathbb{k}$.
- For a fixed $\mathbb{k}$, the Cohen–Macaulayness of L is a topological property: it depends only on the homeomorphism type of L.
- For $\sigma = \emptyset$, the condition means that $\widetilde{H}^{\bullet}(L, \mathbb{Z})$ is concentrated in degree n; it also implies that L is pure, i.e., all its maximal simplices have dimension n.

Theorem (N. Brady–Meier 2001, Jensen–Meier 2005)

A RAAG G_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, G_Γ is a Poincaré duality group if and only if Γ is a complete graph.

Theorem (DSY17)

A toric complex T_L is an abelian duality space (of dimension $\dim L + 1$) if and only if L is Cohen-Macaulay, in which case both the resonance and characteristic varieties of T_L propagate.

Corollary

Let L be a Cohen–Macaulay complex over $\Bbbk$. Suppose there is a subset $\mathsf{W} \subset \mathsf{V}$ of the vertex set and a simplex σ supported on $\mathsf{V} \setminus \mathsf{W}$ such that

$$\widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \Bbbk) \neq 0,$$

for some $i \geqslant |\sigma|$. Then, for all $i \leqslant j \leqslant \dim(L) + 1$, there exists a subset $\mathsf{W} \subset \mathsf{W}' \subset \mathsf{V}$ and a simplex σ' supported on $\mathsf{V} \setminus \mathsf{W}'$ such that

$$\widetilde{H}_{j-1-|\sigma'|}(\mathrm{lk}_{L_{\mathsf{W}'}}(\sigma'), \Bbbk) \neq 0.$$

- Is there a direct, combinatorial proof of this result?
- Is there a simplicial complex L which is not Cohen–Macaulay but for which the resonance varieties of T_L still propagate?

Bestvina–Brady groups

- The Bestvina–Brady group associated to a graph Γ is defined as $N_\Gamma = \ker(\nu\colon G_\Gamma \to \mathbb{Z})$, where $\nu(v) = 1$, for each $v \in V(\Gamma)$.
- Meier–VanWyck 1995: N_Γ is finitely generated iff Γ is connected.
- Bestvina–Brady 1997: N_Γ is finitely presented iff the flag complex Δ_Γ is simply connected.
- In this case, an explicit finite presentation for N_Γ (with generators the edges of Γ) was given by Dicks–Leary 1999.
- E.g:
  - If $\Gamma = \overline{K}_2$, then $G_\Gamma = F_2$ and $N_\Gamma = F_\infty$.
  - If $\Gamma = K_{2,2}$, then $G_\Gamma = F_2 \times F_2$ and N_Γ is finitely generated but not finitely presented, since $H_2(N_\Gamma; \mathbb{Z}) = \mathbb{Z}^\infty$ (Stallings 1963).
  - If Γ is a tree on n vertices, then $N_\Gamma = F_{n-1}$.

- BB97: A counterexample to either the Eilenberg–Ganea conjecture or the Whitehead asphericity conjecture can be constructed from N_Γ, where Γ is the 1-skeleton of a triangulation of the Poincaré sphere.
- (Papadima–S. 2007): if Γ is connected, then $\mathbb{Z}$ acts trivially on $H_1(N_\Gamma, \mathbb{Z})$.
- The cohomology ring $H^*(N_\Gamma, \Bbbk)$ was computed in PS07 and by Leary–Saadetoğlu (2011).
- The jump loci $\mathcal{R}^1_1(N_\Gamma, \Bbbk)$ and $\mathcal{V}^1_1(N_\Gamma, \Bbbk)$ were computed in PS07.

THEOREM (DAVIS–OKUN 2012)

Suppose Δ_Γ is acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay.

THEOREM (DSY18)

A Bestvina–Brady group N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen–Macaulay.