# Topology and geometry of cohomology jump loci

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# SUPPORT LOCI

- Let $\Bbbk$ be an algebraically closed field.
- Let $S$ be a commutative, finitely generated $\Bbbk$-algebra.
- Let $\operatorname{Spec}(S) = \operatorname{Hom}_{\Bbbk\text{-alg}}(S, \Bbbk)$ be the maximal spectrum of $S$.
- Let $E\colon \cdots \to E_i \xrightarrow{d_i} E_{i-1} \to \cdots \to E_0 \to 0$ be an $S$-chain complex.
- The *support varieties* of $E$ are the subsets of $\operatorname{Spec}(S)$ given by

$$\mathcal{W}^i_d(E) = \operatorname{supp}\Big(\bigwedge^d H_i(E)\Big).$$

- They depend only on the chain-homotopy equivalence class of $E$.
- For each $i \geqslant 0$, $\operatorname{Spec}(S) = \mathcal{W}^i_0(E) \supseteq \mathcal{W}^i_1(E) \supseteq \mathcal{W}^i_2(E) \supseteq \cdots$.
- If all $E_i$ are finitely generated $S$-modules, then the sets $\mathcal{W}^i_d(E)$ are Zariski closed subsets of $\operatorname{Spec}(S)$.

# HOMOLOGY JUMP LOCI

- The *homology jump loci* of the $S$-chain complex $E$ are defined as

$$\mathcal{V}^i_d(E) = \{\mathfrak{m} \in \operatorname{Spec}(S) \mid \dim_{\Bbbk} H_i(E \otimes_S S/\mathfrak{m}) \geqslant d\}.$$

- They depend only on the chain-homotopy equivalence class of $E$.
- For each $i \geqslant 0$, $\operatorname{Spec}(S) = \mathcal{V}^i_0(E) \supseteq \mathcal{V}^i_1(E) \supseteq \mathcal{V}^i_2(E) \supseteq \cdots$.
- (Papadima–S. 2013) Suppose $E$ is a chain complex of *free*, finitely generated $S$-modules. Then,
  - Each $\mathcal{V}^i_d(E)$ is a Zariski closed subset of $\operatorname{Spec}(S)$.
  - For each $q$,

$$\bigcup_{i \leqslant q} \mathcal{V}^i_1(E) = \bigcup_{i \leqslant q} \mathcal{W}^i_1(E).$$

# RESONANCE VARIETIES

- Let $A$ be a commutative graded $\Bbbk$-algebra, with $A^0 = \Bbbk$.
- Let $a \in A^1$, and assume $a^2 = 0$ (this condition is redundant if $\operatorname{char}(\Bbbk) \neq 2$, by graded-commutativity of the multiplication in $A$).
- The *Aomoto complex* of $A$ (with respect to $a \in A^1$) is the cochain complex of $\Bbbk$-vector spaces,

$$(A, a)\colon \; A^0 \xrightarrow{\;a\;} A^1 \xrightarrow{\;a\;} A^2 \xrightarrow{\;a\;} \cdots,$$

  with differentials given by $b \mapsto a \cdot b$, for $b \in A^i$.
- The *resonance varieties* of $A$ are the sets

$$\mathcal{R}^i_d(A) = \{a \in A^1 \mid a^2 = 0 \text{ and } \dim_{\Bbbk} H^i(A, a) \geqslant d\}.$$

- If $A$ is locally finite (i.e., $\dim_{\Bbbk} A^i < \infty$, for all $i \geqslant 1$), then the sets $\mathcal{R}^i_d(A)$ are Zariski closed cones inside the affine space $A^1$.

# CHARACTERISTIC VARIETIES

- Let $X$ be a connected, finite-type CW-complex.
- Fundamental group $\pi = \pi_1(X, x_0)$: a finitely generated, discrete group, with $\pi_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.
- Fix a field $\Bbbk$ with $\overline{\Bbbk} = \Bbbk$, and let $S = \Bbbk[\pi_{\mathrm{ab}}]$.
- Identify $\operatorname{Spec}(S)$ with the character group $\widehat{\pi_{\mathrm{ab}}} = \hat{\pi} = \operatorname{Hom}(\pi, \Bbbk^*)$.
- The characteristic varieties of $X$ are the homology jump loci of free $S$-chain complex $E = C_*(X^{\mathrm{ab}}, \Bbbk)$:

$$\mathcal{V}^i_d(X, \Bbbk) = \{\rho \in \hat{\pi} \mid \dim_{\mathbb{C}} H_i(X, \Bbbk_\rho) \geqslant d\}.$$

- Each set $\mathcal{V}^i_d(X, \Bbbk)$ is a subvariety of $\hat{\Bbbk}$.

- *Homotopy invariance:* If $X \simeq Y$, then $\mathcal{V}^i_d(Y,\Bbbk) \cong \mathcal{V}^i_d(X,\Bbbk)$.
- *Product formula:* $\mathcal{V}^i_1(X_1 \times X_2,\Bbbk) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1,\Bbbk) \times \mathcal{V}^q_1(X_2,\Bbbk)$.
- *Degree 1 interpretation:* The sets $\mathcal{V}^1_d(X,\Bbbk)$ depend only on $\pi=\pi_1(X)$—in fact, only on $\pi/\pi''$. Write them as $\mathcal{V}^1_d(\pi,\Bbbk)$.
- *Functoriality:* If $\varphi\colon \pi \twoheadrightarrow G$ is an epimorphism, then $\hat\varphi\colon \widehat{G} \hookrightarrow \widehat{\pi}$ restricts to an embedding $\mathcal{V}^1_d(G,\Bbbk) \hookrightarrow \mathcal{V}^1_d(\pi,\Bbbk)$, for each $d$.
- *Universality:* Given any subvariety $W \subset (\Bbbk^*)^n$, there is a finitely presented group $\pi$ such that $\pi_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(\pi,\Bbbk) = W$.
- *Alexander invariant interpretation:* Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}},\Bbbk)$ as a module over $S=\Bbbk[\pi_{\mathrm{ab}}]$. Then:
$$\bigcup_{j\leqslant i} \mathcal{V}^j_1(X) = \operatorname{supp}\Big(\bigoplus_{j\leqslant i} H_j(X^{\mathrm{ab}},\Bbbk)\Big).$$

# The tangent cone theorem

- The *resonance varieties* of $X$ (with coefficients in $\Bbbk$) are the loci $\mathcal{R}^i_d(X,\Bbbk)$ associated to the cohomology algebra $A = H^*(X,\Bbbk)$.
- Each set $\mathcal{R}^i_d(X) := \mathcal{R}^i_d(X,\mathbb{C})$ is a homogeneous subvariety of $H^1(X,\mathbb{C}) \cong \mathbb{C}^n$, where $n = b_1(X)$.
- Recall that $\mathcal{V}^i_d(X) := \mathcal{V}^i_d(X,\mathbb{C})$ is a subvariety of $H^1(X,\mathbb{C}^*) \cong (\mathbb{C}^*)^n \times \operatorname{Tors}(H_1(X,\mathbb{Z}))$.
- (Libgober 2002) $\operatorname{TC}_1(\mathcal{V}^i_d(X)) \subseteq \mathcal{R}^i_d(X)$.
- Given a subvariety $W \subset H^1(X,\mathbb{C}^*)$, let $\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq \operatorname{TC}_1(W)$.
- Thus, $\tau_1(\mathcal{V}^i_d(X)) \subseteq \operatorname{TC}_1(\mathcal{V}^i_d(X)) \subseteq \mathcal{R}^i_d(X)$.

- $X$ is *formal* if there is a zig-zag of cdga quasi-isomorphisms from $(A_{\mathrm{PL}}(X, \mathbb{Q}), d)$ to $(H^*(X, \mathbb{Q}), 0)$.
- $X$ is $k$*-formal* (for some $k \geqslant 1$) if each of these morphisms induces an iso in degrees up to $k$, and a monomorphism in degree $k+1$.
- $X$ is $1$-formal if and only if $\pi = \pi_1(X)$ is $1$-formal, i.e., its Malcev Lie algebra, $\mathfrak{m}(\pi) = \mathrm{Prim}(\widehat{\mathbb{Q}\pi})$, is quadratic.
- For instance, compact Kähler manifolds and complements of hyperplane arrangements are formal.
- (Dimca–Papadima–S. 2009) Let $X$ be a $1$-formal space. Then, for each $d > 0$,

$$\tau_1(\mathcal{V}^1_d(X)) = \mathrm{TC}_1(\mathcal{V}^1_d(X)) = \mathcal{R}^1_d(X).$$

  Consequently, $\mathcal{R}^1_d(X)$ is a finite union of rationally defined linear subspaces in $H^1(X, \mathbb{C})$.

This theorem yields a very efficient formality test.

### EXAMPLE

Let $\pi = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4]\rangle$. Then $\mathcal{R}_1^1(\pi) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$ splits into linear subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, $\pi$ is *not* 1-formal.

### EXAMPLE

Let $F(\Sigma_g, n)$be the configuration space of $n$ labeled points of a Riemann surface of genus $g$ (a smooth, quasi-projective variety).

Then $\pi_1(F(\Sigma_g, n)) = P_{g,n}$, the pure braid group on $n$ strings on $\Sigma_g$. Compute:

$$\mathcal{R}_1^1(P_{1,n}) = \left\{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n \end{array}\right\}$$

For $n \geqslant 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

# PROPAGATION OF COHOMOLOGY JUMP LOCI

(Denham–S.–Yuzvinsky 2013)

- Assume $X$ is an *abelian duality space* of dimension $n$, i.e., $H^p(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Given a character character $\rho\colon \pi \to \mathbb{C}^*$, if $H^p(X, \mathbb{C}_\rho) \neq 0$, then $H^q(X, \mathbb{C}_\rho) \neq 0$ for all $p \leqslant q \leqslant n$.
- Thus, the characteristic varieties of $X$ "propagate":

$$\mathcal{V}_1^1(X) \subseteq \mathcal{V}_1^2(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X).$$

- Moreover, if $X$ admits a minimal cell structure, then

$$\mathcal{R}_1^1(X) \subseteq \mathcal{R}_1^2(X) \subseteq \cdots \subseteq \mathcal{R}_1^n(X).$$

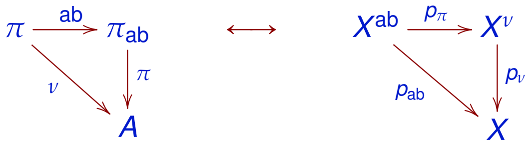
- If $\mathcal{A}$ is an arrangement of rank $d$, then its complement, $M(\mathcal{A})$, is an abelian duality space of dim $d$. Thus, both the characteristic and the resonance varieties of $M(\mathcal{A})$ propagate.

# APPLICATIONS OF COHOMOLOGY JUMP LOCI

- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants
- Obstructions to (quasi-) projectivity
  - Right-angled Artin groups and Bestvina–Brady groups
  - 3-manifold groups, Kähler groups, and quasi-projective groups
- Resonance varieties and representations of Lie algebras
  - Homological finiteness in the Johnson filtration of automorphism groups
- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement
  - Rational homology of smooth, real toric varieties
- Lower central series and Chen Lie algebras
  - The Chen ranks conjecture for arrangements

# FINITENESS PROPERTIES IN ABELIAN COVERS

- Recall $X$ is a connected, finite-type CW-complex, $\pi = \pi_1(X)$.
- Let $A$ be an abelian group (quotient of $\pi_{\text{ab}}$).
- Equivalence classes of Galois $A$-covers of $X$ can be identified with $\operatorname{Epi}(\pi, A) / \operatorname{Aut}(A) \cong \operatorname{Epi}(\pi_{\text{ab}}, A) / \operatorname{Aut}(A)$.




- In particular, Galois $\mathbb{Z}^r$-covers are parametrized by the Grassmannian $\operatorname{Gr}_r(H^1(X, \mathbb{Q}))$, via the correspondence

$$X^\nu \to X \longleftrightarrow P_\nu := \operatorname{im}(\nu^* \colon \mathbb{Q}^r \to H^1(X, \mathbb{Q}))$$

- Goal: Use the cohomology jump loci of $X$ to analyze the geometric and homological finiteness properties of regular $A$-covers of $X$.

# THE BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- Let $\pi$ be a finitely generated group, $n = b_1(\pi)$.
- Let $S(\pi)$ be the unit sphere in $\mathrm{Hom}(\pi, \mathbb{R}) = \mathbb{R}^n$.
- The BNSR-invariants of $\pi$ form a descending chain of open subsets, $S(\pi) \supseteq \Sigma^1(\pi, \mathbb{Z}) \supseteq \Sigma^1(\pi, \mathbb{Z}) \supseteq \cdots$.
- $\Sigma^k(\pi, \mathbb{Z})$ consists of all $\chi \in S(G)$ for which the monoid $\pi_\chi = \{g \in \pi \mid \chi(g) \geqslant 0\}$ is of type $\mathrm{FP}_k$, i.e., there is a projective $\mathbb{Z}\pi$-resolution $P_\bullet \to \mathbb{Z}$, with $P_i$ finitely generated for all $i \leqslant k$.
- The $\Sigma$-invariants control the finiteness properties of normal subgroups $N \triangleleft \pi$ for which $\pi / N$ is free abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(\pi, N) \subseteq \Sigma^k(\pi, \mathbb{Z})$$

  where $S(\pi, N) = \{\chi \in S(\pi) \mid \chi(N) = 0\}$.
- In particular: $\ker(\chi\colon \pi \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(\pi, \mathbb{Z})$.

- More generally, let $X$ be a connected CW-complex with finite $k$-skeleton, for some $k \geqslant 1$.
- Let $\pi = \pi_1(X, x_0)$. For each $\chi \in S(X) := S(\pi)$, set

$$\widehat{\mathbb{Z}\pi}_\chi = \{\lambda \in \mathbb{Z}^\pi \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R}\}$$

be the Novikov-Sikorav completion of $\mathbb{Z}\pi$.

- Define $\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}\pi}_{-\chi}) = 0, \ \forall\, i \leqslant q\}$.
- (Bieri) If $\pi$ is $\text{FP}_k$, then $\Sigma^q(\pi, \mathbb{Z}) = \Sigma^q(K(\pi, 1), \mathbb{Z})$, $\forall q \leqslant k$.
- The sphere $S(\pi)$ parametrizes all regular, free abelian covers of $X$. The $\Sigma$-invariants of $X$ keep track of the geometric finiteness properties of these covers.

# AN UPPER BOUND FOR THE Σ-INVARIANTS

- Let $\chi \in S(X)$, and set $\Gamma = \operatorname{im}(\chi)$; then $\Gamma \cong \mathbb{Z}^r$, for some $r \geqslant 1$.
- A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is $\chi$-monic if the greatest element in $\chi(\operatorname{supp}(p))$ is $0$, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the Novikov ring: the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all $\chi$-monic polynomials (it's a PID).
- Let $b_i(X,\chi) = \operatorname{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$ be the Novikov-Betti numbers.
- (Papadima–S. 2010) Let $\mathcal{V}^k(X) = \bigcup_{i\leqslant k} \mathcal{V}^i_1(X)$. Then,
  - $-\chi \in \Sigma^k(X,\mathbb{Z}) \implies b_i(X,\chi) = 0, \forall i \leqslant k$.
  - $\chi \notin \tau_1^{\mathbb{R}}(\mathcal{V}^k(X)) \iff b_i(X,\chi) = 0, \forall i \leqslant k$.

  Thus, $\Sigma^i(X,\mathbb{Z}) \subseteq S(X) \backslash S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$.
- In particular, $\Sigma^i(X,\mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

# THE DWYER–FRIED INVARIANTS

- The *Dwyer–Fried invariants* of $X$ are the subsets

$$\Omega^i_r(X) = \{P_\nu \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid b_j(X^\nu) < \infty \text{ for } j \leqslant i\}.$$

- (Dwyer–Fried 1987, Papadima–S. 2010)

$$\Omega^i_r(X) = \{P \in \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \mid \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{V}^i(X)\right) = 0\}.$$

- More generally, for any abelian group $A$, define

$$\Omega^i_A(X) = \{[\nu] \in \mathrm{Epi}(\pi, A) / \mathrm{Aut}(A) \mid b_j(X^\nu) < \infty, \text{ for } j \leqslant i\}.$$

- (S.–Yang–Zhao 2012)

$$\Omega^i_A(X) = \{\, [\nu] \in \mathrm{Epi}(\pi_1(X), A) / \mathrm{Aut}(A) \mid \mathrm{im}(\hat{\nu}) \cap \mathcal{V}^i(X) \text{ is finite} \,\}.$$

- Let $V$ be a homogeneous variety in $\Bbbk^n$. The set $\sigma_r(V) = \big\{P \in \operatorname{Gr}_r(\Bbbk^n) \bigm| P \cap V \neq \{0\}\big\}$ is Zariski closed.
- If $L \subset \Bbbk^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by $L$. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.
- (S. 2013) $\quad \Omega^i_r(X) \subseteq \operatorname{Gr}_r(H^1(X,\mathbb{Q})) \backslash \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X))\big)$.
- Thus, each set $\Omega^i_r(X)$ is contained in the complement of a finite union of special Schubert varieties.
- If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.
- (SYZ) Similar inclusions hold for the sets $\Omega^i_A(X)$, but things get more complicated.

# COMPARING THE $\Sigma$- AND $\Omega$-BOUNDS

- Theorem (S. 2012) If $\Sigma^i(X,\mathbb{Z}) = S(X)\backslash S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))$, then $\Omega^i_r(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q}))\backslash\sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{V}^i(X)))$, for all $r \geqslant 1$.
- Corollary. Suppose there is an integer $r \geqslant 2$ such that $\Omega^i_r(X)$ is *not* Zariski open. Then $\Sigma^i(X,\mathbb{Z}) \subsetneqq S(\tau_1^{\mathbb{R}}(\mathcal{V}^i(X)))^{\complement}$.
- Application. There exist arrangements $\mathcal{A}$ for which the inclusion $\Sigma^1(M(\mathcal{A}),\mathbb{Z}) \subseteq S(\mathcal{R}^1(M(\mathcal{A}),\mathbb{R}))^{\complement}$ is strict.
- On the other hand, if $\mathcal{A}$ is the braid arrangement in $\mathbb{C}^n$, with $\pi_1(M(\mathcal{A})) = P_n$, then equality holds (Koban–McCammond–Meier 2013).
- For more on Novikov homology/BNSR invariants of arrangements, see (Kohno–Pajitnov 2011/13) and (Denham–S.–Yuzvinsky).

- (Delzant 2010/ PS 2010) Let $M$ be a compact Kähler manifold with $b_1(M) > 0$. Then

$$\Sigma^1(M, \mathbb{Z}) = S(\mathcal{R}^1(M))^{\complement}$$

if and only if there is no pencil $f\colon M \to E$ onto an elliptic curve $E$ such that $f$ has multiple fibers.

- (S. 2013) Let $M$ be a compact Kähler manifold.
  - If $M$ admits an orbifold fibration with base genus $g \geqslant 2$, then $\Omega^1_r(M) = \emptyset$, for all $r > b_1(M) - 2g$.
  - Otherwise, $\Omega^1_r(M) = \mathrm{Gr}_r(H^1(M, \mathbb{Q}))$, for all $r \geqslant 1$.
  - Suppose $M$ admits an orbifold fibration with multiple fibers and base genus $g = 1$. Then $\Omega^1_2(M)$ is *not* an open subset of $\mathrm{Gr}_2(H^1(M, \mathbb{Q}))$.

# 3-MANIFOLD GROUPS & KÄHLER GROUPS

- Question (Donaldson–Goldman 1989, Reznikov 1993): Which 3-manifold groups are Kähler groups?
- Reznikov (2002) gave a partial solution.
- Theorem (Dimca–S. 2009) Let $\pi$ be the fundamental group of a closed 3-manifold. Then $\pi$ is a Kähler group $\iff$ $\pi$ is a finite subgroup of $O(4)$, acting freely on $S^3$.
- Idea: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds:
  - Let $M$ be a closed, orientable 3-manifold. Then $H^1(M, \mathbb{C})$ is not 1-isotropic. Moreover, if $b_1(M)$ is even, then $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$.
  - Let $M$ be a compact Kähler manifold with $b_1(M) \neq 0$. If $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$, then $H^1(M, \mathbb{C})$ is 1-isotropic.

- This result can be extended, by allowing the $3$-manifold to have toroidal boundary.
- Theorem (Friedl–S. 2013) Let $N$ be a $3$-manifold with non-empty, toroidal boundary. If $\pi_1(N)$ is a Kähler group, then $N \cong S^1 \times S^1 \times [0,1]$.
- A key ingredient in the proof is a refinement of a result from (Dimca–Papadima–S. 2008): If $\pi$ is a Kähler group, then the Alexander polynomial of $\pi$ is constant.
- Further improvements have been obtained since then by Kotschick and by Biswas, Mj, and Seshadri.

## 3-MANIFOLD GROUPS & QUASI-PROJECTIVE GROUPS

- Theorem (Dimca–Papadima–S. 2011). Let $\pi = \pi_1(N)$, where $N$ is a closed, orientable 3-manifold, and $\pi$ is 1-formal. TFAE:
  - $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(X))$, for some smooth, quasi-projective variety $X$.
  - $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(M))$, where $M = S^3$, $\#^n S^1 \times S^2$, or $S^1 \times \Sigma_g$.
- Theorem (Friedl–S. 2013) Let $N$ be a compact 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all the prime components of $N$ are graph manifolds.
- Again, we use a refinement of a result from (DPS 2008): If $\pi$ is a quasi-projective group, and $b_1(\pi) > 2$, then the Newton polytope of the Alexander polynomial of $\pi$ is a line segment.
- This refinement relies on work of (Artal–Cogolludo-Matei 2013).
- We also use recent, deep results of Agol, Kahn–Markovic, Przytycki–Wise, and Wise on the topology of 3-manifolds that complete the Thurston program.

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