Algebraic invariants of pure braid-like groups

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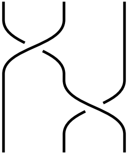
ARTIN'S BRAID GROUPS



- Let B_n be the group of braids on n strings (under concatenation).
- B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.
- Let $P_n = \ker(B_n \twoheadrightarrow S_n)$ be the pure braid group on n strings.
- P_n is generated by $A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ $(1 \leqslant i < j \leqslant n)$.

- $B_n = \mathrm{Mod}^1_{0,n}$, the mapping class group of D^2 with n marked points.

- Thus, B_n is a subgroup of $\mathrm{Aut}(F_n)$. In fact:

$$B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = w x_{\tau(i)} w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n\}.$$

- P_n is a subgroup of $\mathrm{IA}_n = \{\varphi \in \mathrm{Aut}(F_n) \mid \varphi_* = \mathrm{id} \text{ on } H_1(F_n)\}$.

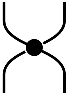
- A classifying space for P_n is the configuration space

$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

- Thus, $B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C})/S_n)$.

- Moreover, $P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, where $\alpha_n \colon P_n \subset B_n \hookrightarrow \mathrm{Aut}(F_n)$.

WELDED BRAID GROUPS



- The set of all permutation-conjugacy automorphisms of F_n forms a subgroup of $wB_n \subset \mathrm{Aut}(F_n)$, called the welded braid group.
- Let $wP_n = \ker(wB_n \twoheadrightarrow S_n) = \mathrm{IA}_n \cap wB_n$ be the pure welded braid group wP_n.
- McCool (1986) gave a finite presentation for wP_n. It is generated by the automorphisms α_{ij} ($1 \leqslant i \neq j \leqslant n$) sending $x_i \mapsto x_j x_i x_j^{-1}$ and $x_k \mapsto x_k$ for $k \neq i$, subject to the relations

$$\begin{aligned} &\alpha_{ij}\alpha_{ik}\alpha_{jk} = \alpha_{jk}\alpha_{ik}\alpha_{ij} && \text{for } i, j, k \text{ distinct,} \\ &[\alpha_{ij}, \alpha_{st}] = 1 && \text{for } i, j, s, t \text{ distinct,} \\ &[\alpha_{ik}, \alpha_{jk}] = 1 && \text{for } i, j, k \text{ distinct.} \end{aligned}$$

- The group wB_n (respectively, wP_n) is the fundamental group of the space of untwisted flying rings (of unequal diameters), cf. Brendle and Hatcher (2013).
- The upper pure welded braid group (or, upper McCool group) is the subgroup $wP_n^+ \subset wP_n$ generated by α_{ij} for $i < j$.
- We have $wP_n^+ \cong F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$.

LEMMA (S.–WANG)

For $n \geqslant 4$, the inclusion $wP_n^+ \hookrightarrow wP_n$ admits no splitting.

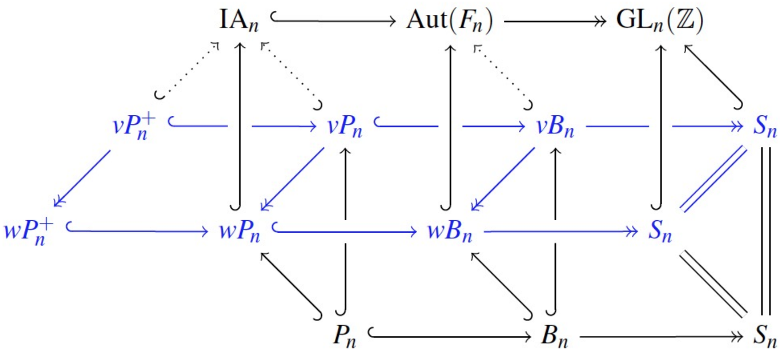
VIRTUAL BRAID GROUPS

- The virtual braid group vB_n is obtained from wB_n by omitting certain commutation relations.
- Let $vP_n = \ker(vB_n \to S_n)$ be the pure virtual braid group.
- Bardakov (2004) gave a presentation for vP_n, with generators x_{ij} ($1 \leqslant i \neq j \leqslant n$), subject to the relations

$$\begin{aligned} x_{ij}x_{ik}x_{jk} &= x_{jk}x_{ik}x_{ij}, && \text{for } i, j, k \text{ distinct,} \\ [x_{ij}, x_{st}] &= 1, && \text{for } i, j, s, t \text{ distinct.} \end{aligned}$$

- Let vP_n^+ be the subgroup of vP_n generated by x_{ij} for $i < j$. The inclusion $vP_n^+ \hookrightarrow vP_n$ *is* a split injection.
- Bartholdi, Enriquez, Etingof, and Rains (2006) studied vP_n and vP_n^+ as groups arising from the Yang-Baxter equation.
- They constructed classifying spaces by taking quotients of permutahedra by suitable actions of the symmetric groups.

SUMMARY OF BRAID-LIKE GROUPS



COHOMOLOGY RINGS AND BETTI NUMBERS

- Arnol'd (1969): $H^*(P_n) = \bigwedge_{i<j}(e_{ij})/\langle e_{jk}e_{ik} - e_{ij}(e_{ik} - e_{jk})\rangle$.
- Jensen, McCammond, and Meier (2006):
  $H^*(wP_n) = \bigwedge_{i\neq j}(e_{ij})/\langle e_{ij}e_{ji}, e_{jk}e_{ik} - e_{ij}(e_{ik} - e_{jk})\rangle$.
- F. Cohen, Pakhianathan, Vershinin, and Wu (2007):
  $H^*(wP_n^+) = \bigwedge_{i<j}(e_{ij})/\langle e_{ij}(e_{ik} - e_{jk})\rangle$.
- Bartholdi et al (2006), P. Lee (2013):
  $H^*(vP_n) = \bigwedge_{i\neq j}(e_{ij})/\langle e_{ij}e_{ji}, e_{ij}(e_{ik} - e_{jk}), e_{ji}e_{ik} = (e_{ij} - e_{ik})e_{jk}\rangle$,
  $H^*(vP_n^+) = \bigwedge_{i<j}(e_{ij})/\langle e_{ij}(e_{ik} - e_{jk}), (e_{ij} - e_{ik})e_{jk}\rangle$.
- All these $\mathbb{Q}$-algebras A are quadratic. In fact, they are all Koszul algebras ($\operatorname{Tor}_i^A(\mathbb{Q},\mathbb{Q})_j = 0$ for $i \neq j$), except for $H^*(wP_n)$, $n \geqslant 4$.
  - P_n: Kohno (1987).
  - wP_n: Conner and Goetz (2015).
  - wP_n^+: D. Cohen and G. Pruidze (2008).
  - vP_n and vP_n^+: Bartholdi et al (2006), Lee (2013).

The Betti numbers of the pure-braid like groups are given by

	P_n	wP_n	wP_n^+	vP_n	vP_n^+
b_i	$s(n, n-i)$	$\binom{n-1}{i} n^i$	$s(n, n-i)$	$L(n, n-i)$	$S(n, n-i)$

Here $s(n, k)$ are the Stirling numbers of the first kind, $S(n, k)$ are the Stirling numbers of the second kind, and $L(n, k)$ are the Lah numbers.

ASSOCIATED GRADED LIE ALGEBRAS

- The *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
- The group commutator induces a graded Lie algebra structure on $\mathrm{gr}(G) = \bigoplus_{k\geqslant 1} (\gamma_k G/\gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{Q}$
- Assume G is finitely generated. Then $\mathrm{gr}(G)$ is also finitely generated: in degree 1, by $\mathrm{gr}_1(G) = H_1(G, \mathbb{Q})$.
- Let $A^* = H^*(G, \mathbb{Q})$, let $\mu_A \colon A^1 \wedge A^1 \to A^2$ be the cup-product map, and $\mu_A^{\vee} \colon A_2 \to A_1 \wedge A_1$ its dual, where $A_i = (A^i)^{\vee}$.
- Define the *holonomy Lie algebra* $\mathfrak{h}(G) := \mathfrak{h}(A)$ as the quotient $\mathrm{Lie}(A_1)$ by the ideal generated by $\mathrm{im}(\mu_A^{\vee}) \subset A_1 \wedge A_1 = \mathrm{Lie}_2(A_1)$.
- There is a canonical surjection $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ which is an isomorphism precisely when $\mathrm{gr}(G)$ is quadratic.

- Let $\phi_k(G) = \dim \operatorname{gr}_k(G)$ be the *LCS ranks* of G.
- E.g.: $\phi_k(F_n) = \frac{1}{k}\sum_{d\mid k} \mu(\frac{k}{d}) n^d$.
- By the Poincaré–Birkhoff–Witt theorem,

$$\prod_{k=1}^{\infty}(1-t^k)^{-\phi_k(G)} = \operatorname{Hilb}(U(\operatorname{gr}(G)), t).$$

PROPOSITION (PAPADIMA–YUZVINSKY 1999)

Suppose $\operatorname{gr}(G)$ *is quadratic and* $A = H^*(G;\mathbb{Q})$ *is Koszul. Then* $\operatorname{Hilb}(U(\operatorname{gr}(G)), t)\cdot \operatorname{Hilb}(A, -t) = 1$.

- Let G be a pure braid-like group. Then $\operatorname{gr}(G)$ is quadratic.
- Furthermore, if $G \neq wP_n$ ($n \geqslant 4$), then $H^*(G;\mathbb{Q})$ is Koszul.
- Thus,

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k(G)} = \sum_{i\geqslant 0} b_i(G)(-t)^i.$$

Chen Lie algebras

- The *Chen Lie algebra* of a f.g. group G is $\operatorname{gr}(G/G'')$, the associated graded Lie algebra of its maximal metabelian quotient.
- Let $\theta_k(G) = \dim \operatorname{gr}_k(G/G'')$ be the *Chen ranks* of G.
- Easy to see: $\theta_k(G) \leqslant \phi_k(G)$ and $\theta_k(G) = \phi_k(G)$ for $k \leqslant 3$.
- K.-T. Chen(1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.

Theorem (D. Cohen–S. 1993)

The Chen ranks $\theta_k = \theta_k(P_n)$ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and $\theta_k = (k-1)\binom{n+1}{4}$ for $k \geqslant 3$.

Corollary

Let $\Pi_n = F_{n-1} \times \cdots \times F_1$. Then $P_n \not\cong \Pi_n$ for $n \geqslant 4$, although both groups have the same Betti numbers and LCS ranks.

THEOREM (D. COHEN–SCHENCK 2015)

$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$, *for* $k \gg 0$.

THEOREM (S.–WANG)

The Chen ranks $\theta_k = \theta_k(wP_n^+)$ *are given by* $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, *and*

$$\theta_k = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \text{ for } k \geqslant 3.$$

COROLLARY

$wP_n^+ \not\cong P_n$ *and* $wP_n^+ \not\cong \Pi_n$ *for* $n \geqslant 4$, *although all three groups have the same Betti numbers and LCS ranks.*

This answers a question of F. Cohen et al. (2007).

RESONANCE VARIETIES

- Let A be a graded $\mathbb{C}$-algebra with $A^0 = \mathbb{C}$ and $\dim A^1 < \infty$.
- The (first) *resonance variety* of A is defined as

$$\mathcal{R}_1(A) = \{a \in A^1 \mid \exists b \in A^1 \backslash \mathbb{C} \cdot a \text{ such that } a \cdot b = 0 \in A^2\}.$$

- For a finitely generated group G, define $\mathcal{R}_1(G) := \mathcal{R}_1(H^*(G; \mathbb{C}))$.
- For instance, $\mathcal{R}_1(F_n) = \mathbb{C}^n$ for $n \geqslant 2$, and $\mathcal{R}_1(\mathbb{Z}^n) = \{0\}$.

PROPOSITION (D. COHEN–S. 1999)

$\mathcal{R}_1(P_n)$ is a union of $\binom{n}{3} + \binom{n}{4}$ linear subspaces of dimension 2.

PROPOSITION (D. COHEN 2009)

$\mathcal{R}_1(wP_n)$ is a union of $\binom{n}{2}$ linear subspaces of dimension 2 and $\binom{n}{3}$ linear subspaces of dimension 3.

PROPOSITION (S.–WANG)

$$\mathcal{R}_1(wP_n^+) = \bigcup_{2\leqslant i<j\leqslant n} L_{ij},$$

where L_{ij} is a linear subspace of dimension i.

LEMMA (S.–WANG)

$\mathcal{R}_1(vP_4^+)$ is the subvariety of $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ defined by

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,$$
$$x_{12}(x_{13}x_{14}-x_{23}x_{24}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0.$$

FORMALITY PROPERTIES

- (Quillen 1968) The *Malcev Lie algebra* of a group G is
$$\mathfrak{m}(G) = \mathrm{Prim}(\widehat{\mathbb{Q}G}),$$
the primitives in the I-adic completion of the group algebra of G.
- This is a complete, filtered Lie algebra with $\mathrm{gr}(\mathfrak{m}(G)) \cong \mathrm{gr}(G)$.
- A f.g. group G is 1*-formal* if its Malcev Lie algebra is quadratic.
- Thus, if G is 1-formal, then G is *graded-formal*, i.e., $\mathrm{gr}(G)$ is quadratic.
- Conversely, if G is graded-formal and *filtered-formal*, i.e., $\mathfrak{m}(G) \cong \widehat{\mathrm{gr}(\mathfrak{m}(G))}$, then G is 1-formal.
- Formality properties are preserved under (finite) direct products and free products, and under split injections.

THEOREM (DIMCA–PAPADIMA–S. 2009)

If G is 1-formal, then $\mathcal{R}_1(G)$ is a union of projectively disjoint, rationally defined linear subspaces of $H^1(G,\mathbb{C})$.

THEOREM (KOHNO 1983)

Fundamental groups of complements of complex projective hypersurfaces (e.g., F_n and P_n) are 1-formal.

THEOREM (BERCEANU–PAPADIMA 2009)

wP_n and wP_n^+ are 1-formal.

THEOREM (S.–WANG)

vP_n and vP_n^+ are 1-formal if and only if $n \leqslant 3$.

PROOF.

- There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \dots
\end{array}$$

- $vP_2^+ = \mathbb{Z}$ and $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Thus, they are both 1-formal.
- $vP_3 \cong N * \mathbb{Z}$ and $P_4 \cong N \times \mathbb{Z}$. Thus, vP_3 is 1-formal.
- $\mathcal{R}_1(vP_4^+)$ is non-linear. Thus, vP_4^+ is not 1-formal.
- Hence, vP_n^+ and vP_n ($n \geqslant 4$) are also not 1-formal.

□

FORMALITY AND CHEN LIE ALGEBRAS

THEOREM (S–WANG)

Let G be a finitely generated group. The quotient map $G \twoheadrightarrow G/G''$ induces a natural epimorphism of graded Lie algebras,

$$\operatorname{gr}(G)/\operatorname{gr}(G)'' \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G'')\,.$$

Moreover, if G is filtered-formal, this map is an isomorphism.

THEOREM (PAPADIMA–S 2004, S–WANG)

There is a natural epimorphism of graded Lie algebras,

$$\mathfrak{h}(G)/\mathfrak{h}(G)'' \longrightarrow\!\!\!\!\!\rightarrow \operatorname{gr}(G/G'')\,.$$

Moreover, if G is 1-formal, then this map is an isomorphism.

- Hence, if $A = H^*(G,\mathbb{Q})$, and $\theta_k(A) := \dim \mathfrak{h}(A)/\mathfrak{h}(A)''$, then $\theta_k(A) \geqslant \theta_k(G)$, with equality if G is 1-formal.

The resonance Chen ranks formula

Conjecture (S. 2001)

Let G be a hyperplane arrangement group. Let $c_m(G)$ be the number of m-dimensional components of $\mathcal{R}_1(G)$. Then, for $k \gg 1$,

$$\theta_k(G) = \sum_{m \geqslant 2} c_m(G) \cdot \binom{m+k-2}{k}.$$

- The conjecture was based in part on $\theta_k(P_n)$ versus $\mathcal{R}_1(P_n)$.
- The inequality $\geqslant$ was proved in [Schenck–S, 2006], using the 1-formality of arrangement groups.

Theorem (D. Cohen–Schenck 2015)

More generally, the conjecture holds if G is a 1-formal, commutator-relators group for which the components of $\mathcal{R}_1(G)$ are isotropic, projectively disjoint, and reduced (as schemes).

THEOREM (S.–WANG)

Let A be a graded algebra with $\dim A^1 < \infty$. Suppose that all the irreducible components of the first resonance variety $\mathcal{R}^1(A)$ are linear, isotropic, and pairwise projectively disjoint. Then, for all $k \gg 0$,

$$\theta_k(A) \geqslant (k-1) \sum_{m \geqslant 2} \binom{m+k-2}{k} c_m(A).$$

Furthermore, if each irreducible component of $\mathcal{R}^1(A)$ is reduced, then equality holds for $k \gg 0$.

- For $A = H^*(G, \mathbb{C})$, this theorem recovers that of Cohen and Schenck, without the commutator-relators assumption.
- The groups wP_n satisfy the Chen ranks formula.
- However, wP_n^+ does *not* satisfy the Chen ranks formula for $n \geqslant 4$. (The components of $\mathcal{R}_1(wP_n^+)$ are linear and projectively disjoint, but they are neither isotropic, nor reduced).

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