Lie algebras associated to hyperplane arrangements

Alexandru Suciu

Northeastern University

Workshop: Various Guises of Reflection Arrangements

International Centre for Mathematical Sciences, Edinburgh, UK

March 14, 2023

Lower central series

- Let G be a group. The *lower central series* $\{\gamma_k(G)\}_{k\geqslant 1}$ is defined inductively by $\gamma_1(G) = G$ and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \lhd G$, then $[H, K] \lhd G$.
- The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of G. Moreover, $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- In particular, it is a *central* series, i.e., $[G, \gamma_k(G)] \subseteq \gamma_{k+1}(G)$.
- In fact, it is the fastest descending central series for G.
- It is also a *normal* series, i.e., $\gamma_k(G) \lhd G$. Each quotient,

  $$\mathrm{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  lies in the center of $G/\gamma_{k+1}(G)$, and thus is an abelian group.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \mathrm{gr}_k(G)$.

Associated graded Lie algebra

- For a coefficient ring $\Bbbk$, we let $\mathrm{gr}(G;\Bbbk) = \bigoplus_{k\geqslant 1} \mathrm{gr}_k(G)\otimes \Bbbk$.
- This is a graded Lie algebra, with addition induced by the group multiplication and with Lie bracket $[\,,\,]\colon \mathrm{gr}_k \times \mathrm{gr}_\ell \to \mathrm{gr}_{k+\ell}$ induced by the group commutator.
- The construction is functorial. Write $\mathrm{gr}(G) = \mathrm{gr}(G;\mathbb{Z})$.
- Example: if F_n is the free group of rank n, then
  - $\mathrm{gr}(F_n)$ is the free Lie algebra $\mathsf{Lie}(\mathbb{Z}^n)$.
  - $\mathrm{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k}\sum_{d\mid k}\mu(d)n^{\frac{k}{d}}$.
- $G/\gamma_n(G)$ is the maximal $(n-1)$-step nilpotent quotient of G.
- $G/\gamma_2(G) = G_{\mathrm{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G;\mathbb{Z})$.

Chen Lie algebras

- Let $G^{(i)}$ be the *derived series* of G, starting at $G^{(1)} = G'$, $G^{(2)} = G''$, and defined inductively by $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.
- The quotient groups, $G/G^{(i)}$, are solvable; $G/G' = G_{\mathrm{ab}}$, while G/G'' is the maximal metabelian quotient of G.
- The *i-th Chen Lie algebra* of G is defined as $\mathrm{gr}(G/G^{(i)}; \Bbbk)$.
- The projection $q_i \colon G \twoheadrightarrow G/G^{(i)}$, induces a surjection $\mathrm{gr}_k(G; \Bbbk) \twoheadrightarrow \mathrm{gr}_k(G/G^{(i)}; \Bbbk)$, which is an iso for $k \leqslant 2^i - 1$.
- Assuming G is finitely generated, write $\theta_k(G) = \mathrm{rank}\, \mathrm{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- Example (K.-T. Chen 1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geqslant 2$.

Holonomy Lie algebra

- A quadratic approximation of the Lie algebra $\mathrm{gr}(G;\Bbbk)$, where $\Bbbk$ is a field, is the *holonomy Lie algebra* of G, defined as

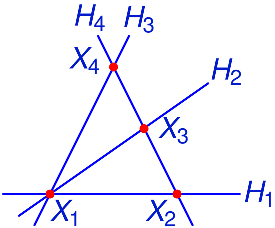
$$\mathfrak{h}(G;\Bbbk) := \mathrm{Lie}(H_1(G;\Bbbk))/\langle \mathrm{im}(\mu_G^\vee)\rangle,$$

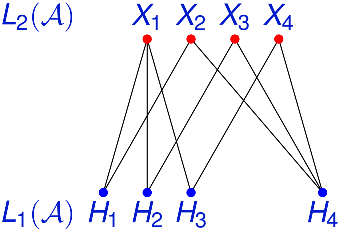
  where
  - $L = \mathrm{Lie}(V)$ the free Lie algebra on the $\Bbbk$-vector space $V = H_1(G;\Bbbk)$, with $L_1 = V$ and $L_2 = V \wedge V$;
  - $\mu_G^\vee \colon H_2(G;\Bbbk) \to V \wedge V$ is the dual of the cup product map $\mu_G \colon H^1(G;\Bbbk) \wedge H^1(G;\Bbbk) \to H^2(G;\Bbbk)$.
- There is natural epimorphism of graded Lie algebras, $\mathfrak{h}(G;\Bbbk) \twoheadrightarrow \mathrm{gr}(G;\Bbbk)$, which restricts to isos in degrees 1 and 2.
- For each $i \geqslant 2$, this morphism factors through epimorphisms $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \twoheadrightarrow \mathrm{gr}(G/G^{(i)};\Bbbk)$.

Malcev Lie algebra

- Let $\Bbbk$ be a field of characteristic 0. Then $\Bbbk G$ is a Hopf algebra, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon\colon \Bbbk G \to \Bbbk$.
- Let $J = \ker \varepsilon$. The J-adic completion $\widehat{\Bbbk G} = \varprojlim_k \Bbbk G/J^k$ is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set $\mathfrak{m}(G;\Bbbk) = \operatorname{Prim}(\widehat{\Bbbk G})$ of all such elements, with bracket $[x, y] = xy - yx$, is the *Malcev Lie algebra* of G.
- If G is finitely generated, then $\operatorname{gr}(\mathfrak{m}(G;\Bbbk)) \cong \operatorname{gr}(G;\Bbbk)$.
- G is *filtered-formal* (over $\Bbbk$), if there is an isomorphism of filtered Lie algebras, $\mathfrak{m}(G;\Bbbk) \cong \widehat{\operatorname{gr}}(G;\Bbbk)$.
- G is 1*-formal* (over $\Bbbk$) if it is filtered formal and the projection $\mathfrak{h}(G;\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk)$ is an isomorphism; that is, $\mathfrak{m}(G;\Bbbk) \cong \widehat{\mathfrak{h}}(G;\Bbbk)$.
- (Papadima–S. 2004) If G is 1-formal, then the maps $\mathfrak{h}(G;\Bbbk)/\mathfrak{h}(G;\Bbbk)^{(i)} \twoheadrightarrow \operatorname{gr}(G/G^{(i)};\Bbbk)$ are isomorphisms.

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^d$.
- For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$; set $f = \prod_{H\in\mathcal{A}} \alpha_H$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement*: $M(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$. It is a smooth, quasi-projective variety and also a Stein manifold. It has the homotopy type of a finite, connected, d-dimensional CW-complex.

Example (The Boolean arrangement)

- $\mathfrak{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathfrak{B}_n)$: Boolean lattice of subsets of $\{0, 1\}^n$.
- $M(\mathfrak{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n \simeq K(\mathbb{Z}^n, 1)$.

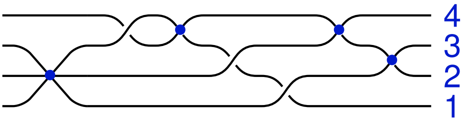
Example (The braid arrangement)

- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: the (ordered) configuration space of n distinct points in $\mathbb{C}$; it is a classifying space $K(P_n, 1)$ for the pure braid group on n strands, P_n.

Cohomology rings of arrangements

- The space $M(\mathcal{A})$ admits a minimal cell structure.
- The groups $H_q(M(\mathcal{A});\mathbb{Z})$ are finitely generated and torsion-free, with ranks given by $\sum_{q=0}^{\ell} b_q(M(\mathcal{A}))t^q=\sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)}$, where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is defined by $\mu(\mathbb{C}^d) = 1$ and $\mu(X) = -\sum_{Y\supsetneq X} \mu(Y)$.
- Let E be the $\mathbb{Z}$-exterior algebra on degree 1 cohomology classes $e_H = \frac{1}{2\pi i}[d\log(\alpha_H)]$ dual to the meridians x_H around $H \in \mathcal{A}$.
- Let $\partial\colon E^* \to E^{*-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_X = \prod_{H\supseteq X} e_H$ for each $X \in \mathcal{L}(A)$.
- Arnold, Brieskorn, Orlik–Solomon showed: $H^*(M(\mathcal{A});\mathbb{Z}) \cong E/I$, where $I = \langle \partial e_X : \operatorname{rank}(X) < |X|\rangle$.
- M. Kim and B. Shapiro: The quasi-projective variety M admits a *pure* mixed Hodge structure.
- Thus, M is $\mathbb{Q}$-formal (albeit not $\mathbb{Z}_p$-formal, in general).

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar section of $\mathcal{A}$. Then the arrangement group, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$. Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$ and the braids $\alpha_r \in P_n$ can be read off an associated braided wiring diagram,



- The group $G(\mathcal{A})$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$.

Holonomy and associated graded Lie algebras

- The holonomy Lie algebra of $G = G(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$,

$$\mathfrak{h}(G) = \mathrm{Lie}(x_H : H \in \mathcal{A}) \Big/ \text{ideal} \left\{ \Big[x_H, \sum_{\substack{K \in \mathcal{A} \\ K \supset Y}} x_K \Big] : \begin{array}{l} H \in \mathcal{A}, Y \in L_2(\mathcal{A}) \\ H \supset Y \end{array} \right\}.$$

- Since M is formal, the group G is 1-formal. Hence, $\mathrm{gr}(G) \otimes \mathbb{Q}$ is determined by $H^{\leqslant 2}(M, \mathbb{Q})$, and thus, by $L_{\leqslant 2}(\mathcal{A})$.
- In fact, the surjection $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ induces an isomorphism, $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$.
- (Papadima–S. 2004) The Chen ranks $\theta_k(G)$ are also determined by $L_{\leqslant 2}(\mathcal{A})$.
- Explicit combinatorial formulas for the LCS ranks $\phi_k(G)$ are known in some cases, but not in general.

- $U(\mathfrak{h}(G)\otimes\mathbb{Q}) = \mathrm{Ext}^1_A(\mathbb{Q},\mathbb{Q}) = \overline{A}^!$, the quadratic dual of the quadratic closure of $A = H^*(M,\mathbb{Q})$.
- (Falk–Randell 1985) If $\mathcal{A}$ is *supersolvable* with exponents $d_1,\dots,d_\ell$, then $\phi_k(G) = \sum_{i=1}^{\ell}\phi_k(F_{d_i})$. (Also follows from Koszulity of $H^*(M,\mathbb{Q})$ and Koszul duality.)
- (Porter–S. 2020) The map $\mathfrak{h}_3(G)\to\mathrm{gr}_3(G)$ is an isomorphism, but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- (S. 2002) The groups $\mathrm{gr}_k(G)$ may have non-zero torsion for $k\gg 0$. E.g., if $G = G(\text{MacLane})$, then $\mathrm{gr}_5(G) = \mathbb{Z}^{87}\oplus\mathbb{Z}_2^4\oplus\mathbb{Z}_3$.
- (S. 2002): Is the torsion in $\mathrm{gr}(G)$ combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2020): Answer: No!
- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 11 triple points and 2 quintuple points, such that $\mathrm{gr}_k(G^+)\cong\mathrm{gr}_k(G^-)$ for $k\leqslant 3$, yet $\mathrm{gr}_4(G^+) = \mathbb{Z}^{211}\oplus\mathbb{Z}_2$ and $\mathrm{gr}_4(G^-) = \mathbb{Z}^{211}$.

Nilpotent quotients

- The quotient $G/\gamma_3(G)$ is determined by $L_{\leqslant 2}(\mathcal{A})$. Indeed, in the central extension,

$$0 \longrightarrow \operatorname{gr}_2(G) \longrightarrow G/\gamma_3(G) \longrightarrow G_{\mathrm{ab}} \longrightarrow 0,$$

  we have $\operatorname{gr}_2(G) = (I^2)^{\vee}$ and the k-invariant $H_2(G_{\mathrm{ab}}) \to \operatorname{gr}_2(G)$ is dual of the inclusion $I^2 \hookrightarrow E^2 = \bigwedge^2 G_{\mathrm{ab}}$.

- (G. Rybnikov 1994): $G/\gamma_4(G)$ is not always determined by $L_{\leqslant 2}(\mathcal{A})$.

- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 15 triple points, such that $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, and therefore $G^+/\gamma_3(G^+) \cong G^-/\gamma_3(G^-)$ and $\operatorname{gr}_3(G^+) \cong \operatorname{gr}_3(G^-)$, but $G^+/\gamma_4(G^+) \not\cong G^-/\gamma_4(G^-)$.

- The difference can be explained in terms of (generalized) Massey triple products over $\mathbb{Z}_3$.

Decomposable arrangements

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$ be the localization of $\mathcal{A}$ at X.
- The inclusions $\mathcal{A}_X \subset \mathcal{A}$ give rise to maps $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$. Restricting to rank 2 flats yields a map

$$j\colon M(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X)\,.$$

- The induced homomorphism on fundamental groups, $j_\sharp$, defines a morphism of graded Lie algebras,

$$\mathfrak{h}(j_\sharp)\colon \mathfrak{h}(G) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}(G_X)\,.$$

Theorem (Papadima–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ is a surjection for each $k \geqslant 3$ and an iso for $k = 2$.

Definition

$\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism.

Example

Let $\mathcal{A}(\Gamma) = \{z_i - z_j = 0 : (i,j) \in \mathsf{E}(\Gamma)\} \subset \mathcal{A}_n$ be a graphic arrangement. Then $\mathcal{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph.

Theorem (Papadima–S. 2006)

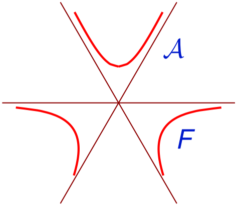
Let $\mathcal{A}$ be a decomposable arrangement, and let $G = G(\mathcal{A})$. Then

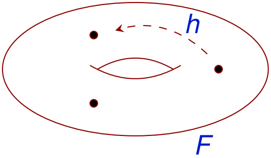
- *The map $\mathfrak{h}'(j_\sharp)\colon \mathfrak{h}'(G) \to \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}'(G_X)$ is an isomorphism of graded Lie algebras.*
- *The map $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism*
- *For each $k \geqslant 2$, the group $\operatorname{gr}_k(G)$ is free abelian of rank $\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})$.*

Theorem (Porter–S. 2020)

Let $\mathcal{A}$ and $\mathfrak{B}$ be decomposable arrangements with $L_{\leqslant 2}(\mathcal{A}) \cong L_{\leqslant 2}(\mathfrak{B})$. Then, for each $k \geqslant 2$,

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathfrak{B})/\gamma_k(G(\mathfrak{B})).$$





- The defining polynomial map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F(\mathcal{A}) := f^{-1}(1)$. The monodromy, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n}z$, where $n = |\mathcal{A}|$.
- F is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension $d-1$ (connected if $d > 1$).
- (Zuber 2010) MHS on F may not be pure and $\pi_1(F)$ may be non-1-formal.

- F is the regular, $\mathbb{Z}_n$-cover of $U = \mathbb{P}(M)$, classified by the epimorphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$, $x_H \mapsto 1$.
- Let $\iota\colon F \hookrightarrow M$ be the inclusion. Induced maps on π_1:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathbb{Z} & & & & \\
 & & & & \downarrow & & & & \\
1 & \longrightarrow & \pi_1(F) & \xrightarrow{\iota_\sharp} & \pi_1(M) & \xrightarrow{f_\sharp} & \mathbb{Z} & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle p_\sharp} & & & & \\
 & & & & \pi_1(U) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}
$$

- $b_1(F) \geqslant n-1$, and may be computed from the characteristic varieties $\mathcal{V}^1_s(U)$. Combinatorial formulas are known in some cases, e.g., if $\mathbb{P}(\mathcal{A})$ has only double or triple points (Papadima–S. 2017).
- (Denham–S. 2016) $H_*(F;\mathbb{Z})$ may have torsion. (Yoshinaga 2020): in fact, $H_1(F;\mathbb{Z})$ may have torsion.

Trivial algebraic monodromy

Theorem (S. 2021)

Suppose $h_* \colon H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Z})$ is the identity. Then

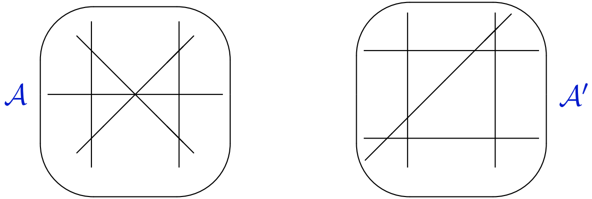
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \cong \operatorname{gr}_{\geqslant 2}(G)$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \cong \operatorname{gr}_{\geqslant 2}(G/G'')$.

Theorem (S. 2021)

Suppose $h_* \colon H_1(F,\mathbb{Q}) \to H_1(F,\mathbb{Q})$ is the identity. Then

- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)) \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G) \otimes \mathbb{Q}$.
- $\operatorname{gr}_{\geqslant 2}(\pi_1(F)/\pi_1(F)'') \otimes \mathbb{Q} \cong \operatorname{gr}_{\geqslant 2}(G/G'') \otimes \mathbb{Q}$.
- $\phi_k(\pi_1(F)) = \phi_k(G)$ *and* $\theta_k(\pi_1(F)) = \theta_k(G)$ *for all* $k \geqslant 2$.

FALK'S PAIR OF ARRANGEMENTS



- Both $\mathcal{A}$ and $\mathcal{A}'$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\mathcal{A}')$. Nevertheless, $M(\mathcal{A}) \simeq M(\mathcal{A}')$.
- Both Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- (S. 2017) $\pi_1(F) \not\cong \pi_1(F')$.
- The difference is picked by the depth-2 characteristic varieties: $\mathcal{V}_2(F) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2(F') = \{1\}$

Yoshinaga's icosidodecahedral arrangement

- The icosidodecahedron is the convex hull of 30 vertices given by the even permutations of $(0, 0, \pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1 + \sqrt{5})/2$.
- It gives rise to an arrangement of 16 hyperplanes in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- $M(\mathcal{A})$ is a $K(G, 1)$.
- $H_1(F; \mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p > 2$), but not over $\mathbb{Z}$.
- Hence, $\operatorname{gr}(\pi_1(F)) \cong \operatorname{gr}(\pi_1(U))$, away from the prime 2. Moreover,
  - $\operatorname{gr}_1(\pi_1(F)) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$
  - $\operatorname{gr}_2(\pi_1(F)) = \mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$
  - $\operatorname{gr}_3(\pi_1(F)) = \mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$
  - $\operatorname{gr}_4(\pi_1(F)) = \mathbb{Z}^{1,405} \oplus \mathbb{Z}_2^{?}$ and $\mathfrak{h}_4(\pi_1(F)) = \mathbb{Z}^{1,405} \oplus \mathbb{Z}_2^{20}$.

References

- Richard D. Porter and Alexander I. Suciu, *Homology, lower central series, and hyperplane arrangements*, European Journal of Mathematics **6** (2020), nr. 3, 1039–1072.
- Alexander I. Suciu, *Lower central series and split extensions*, arXiv:2105.14129.
- Alexander I. Suciu, *Alexander invariants and cohomology jump loci in group extensions*, Annali della Scuola Normale Superiore di Pisa (to appear), doi:10.2422/2036-2145.202112_005, arXiv:2107.05148.