# DIFFERENTIAL GRADED ALGEBRAS, STEENROD CUP-ONE PRODUCTS, BINOMIAL OPERATIONS, AND MASSEY PRODUCTS 

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#### Abstract

Motivated by the construction of Steenrod cup- $i$ products in the singular cochain algebra of a space and in the algebra of non-commutative differential forms, we define a category of binomial cup-one differential graded algebras over the integers and over prime fields of positive characteristic. The Steenrod and Hirsch identities bind the cup-product, the cup-one product, and the differential in a package that we further enhance with a binomial ring structure arising from a ring of integer-valued rational polynomials. This structure allows us to define the free binomial cup-one differential graded algebra generated by a set and derive its basic properties. It also provides the context for defining restricted triple Massey products, which have a smaller indeterminacy than the classical ones, and hence, give stronger homotopy type invariants.


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## 1. Introduction

1.1. Algebraic models for spaces. From its beginnings, algebraic topology has relied on algebraic models to capture topological properties of spaces. Some of the earliest such models are the cochain algebra of a simplicial complex and the de Rham algebra of smooth differential forms on a smooth manifold. The first construction was extended to cellular cochains on a CW-complex and to singular cochains on an arbitrary topological space. The latter construction evolved into Sullivan's algebra of piecewise polynomial forms on a topological space, which provides a natural setting for rational homotopy theory, [34].
An important distinction in this context is between the commutative differential graded algebra (cdga) models such as the de Rham and Sullivan algebras and the non-commutative differential graded algebra (dga) models such as the aforementioned cochain algebras, as well as the algebras of non-commutative differential forms. Although in some ways more difficult to handle, these dga models contain extra information, exemplified by the Steenrod cup-i products, which provide explicit witnesses to the non-commutativity of the usual cup-products and lead to the Steenrod cohomology operations.

We revisit here some of these classical ideas, many of which go back to the work of Steenrod [33] and Hirsch [17], and define several categories of differential graded algebras (over the integers and over prime fields of positive characteristic) with some extra structure, coming from either the cup-one products, or from a binomial ring structure, or both, bound together by suitable compatibility conditions. This structure allows us to define the free binomial cup-one differential graded algebra generated by a set, which will be developed in [27,28] into a theory of 1-minimal models over the integers and over $\mathbb{Z}_{p}$. This framework also provides the context for defining restricted triple Massey products, which is a particular case of a more general construction that will be developed in [29].
1.2. Differential graded algebras and $\cup_{1}$-products. We now outline the main constructions and results of this paper. We start in section 2 by reviewing some basic notions regarding graded algebras and differential graded algebras, including Massey's definition of higher order products [23]. In section 3 we enhance (differential) graded algebras with
an additional structure, working by analogy with Steenrod's cup-one products in cochain algebras.
We define a graded algebra with cup-one products over a coefficient ring $R$ as a graded $R$-algebra $A$ with cup-one product maps, $\cup_{1}: A^{p} \otimes_{R} A^{1} \rightarrow A^{p}$ for $p=1,2$ such that the left Hirsch identity,

$$
\begin{equation*}
(a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+\left(a \cup_{1} c\right) \cup b, \tag{1.1}
\end{equation*}
$$

is satisfied for all $a, b, c \in A^{1}$, and such that the cup-one map $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ gives the $R$-module $R \oplus A^{1}$ the structure of a commutative ring.
Suppose now that $(A, d)$ is a dga such that the underlying algebra $A$ satisfies the above two conditions, and also comes endowed with a map $\cup_{1}: A^{1} \otimes A^{2} \rightarrow A^{2}$. We then say that $(A, d)$ is a differential graded algebra with $\cup_{1}$-products if the following Steenrod identity is satisfied for all $a, b \in A^{1}$ :

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b . \tag{1.2}
\end{equation*}
$$

Note that by the left Hirsch identity if $d a$ is a sum of cup products then the first three terms in equation (1.2) belong to $D^{2}(A)$, the set of elements in $A^{2}$ that can be written as sums of cup products of elements in $A^{1}$. This raises the question of whether there are conditions under which the term $a \cup_{1} d b$ can also be written as a sum of cup products of elements in $A^{1}$. To this end, we assume there is an operation on $D^{2}(A)$ satisfying $(u \cup v) \circ(w \cup z)=\left(u \cup_{1} w\right) \cup\left(v \cup_{1} z\right)$, and consider the identities

$$
\begin{align*}
a \cup_{1}(b \cup c) & =d a \circ(b \cup c)-(b \cup c) \cup_{1} a,  \tag{1.3}\\
d\left(a \cup_{1} b\right) & =-a \cup b-b \cup a+d a \cup_{1} b+d b \cup_{1} a-d a \circ d b, \tag{1.4}
\end{align*}
$$

for $a, b, c \in A^{1}$ and $d a, d b \in D^{2}(A)$. We call the first one the right Hirsch identity and the second one the $\cup_{1}-d$ formula. If (1.2) and (1.3) hold, then it follows that (1.4) is also satisfied. A differential graded $R$-algebra $(A, d)$ with cup-one products and differential that satisfies the $\cup_{1}-d$ formula is called a $\cup_{1}$-differential graded algebra.
We show in sections 4.4 and 5.4 that the right Hirsch identity, and hence the $U_{1}-d$ formula, holds in cochain algebras and in algebras of non-commutative differential forms, respectively. The property that the $\cup_{1}-d$ formula involves only cup products of elements in $A^{1}$ is a critical component in the proofs in sections 7 and 8 of the properties of free $\cup_{1}$-differential graded algebras.
1.3. Cochain algebras and non-commutative differential forms. We have two basic motivating examples for this definition. The first one, discussed in section 4, is the singular cochain algebra, $A=C^{*}(X ; R)$, of a topological space $X$, with coefficients in a commutative ring $R$. In this case, one can take the map $\cup_{1}$ to be the cup-one product
defined by Steenrod in [33], with the Steenrod and Hirsch identities established in [33] and [17], respectively.

The second motivating example, discussed in section 5, is provided by the algebra of noncommutative differential forms, $\Omega^{*}(A ; R)$, on an $R$-algebra $A$, as constructed by Karoubi in [20]. In this context, the Steenrod identities were established by Battikh [2] and the Hirsch identities by Abbassi [1]. As we shall see in [27], sequences of non-commutative Hirsch extensions also satisfy the aforementioned axioms.
1.4. Binomial cup-one dgas. Inspired by the work of Ekedahl [8], Elliott [9], and others we bring in section 6 the notion of binomial ring and tie it up to the previous examples in order to define binomial cup-one algebras and dgas.

A commutative ring $A$ is a binomial ring if $A$ is torsion-free as a $\mathbb{Z}$-module, and the elements $\binom{a}{n}:=a(a-1) \cdots(a-n+1) / n!$ lie in $A$ for every $a \in A$ and every $n>0$, thereby defining maps $\zeta_{n}: A \rightarrow A, a \mapsto\binom{a}{n}$. Combining the notions of cup-one (differential) graded algebras and binomial rings we arrive at the notion of a binomial cup-one (differential) graded algebra. We define binomial $\cup_{1}$-dgas as graded algebras with cup-one products endowed with a differential satisfying the Leibniz and $\cup_{1}-d$ formulas.
Our goal here is to derive a formula for the differential of $\zeta_{n}(a)$ for $a$ a cocycle in $A^{1}$ and $n \geq 1$, in the setting where $A$ is a binomial $\cup_{1}$-dga. We achieve this in Theorem 6.13, where we show that

$$
\begin{equation*}
d \zeta_{n}(a)=-\sum_{k=1}^{n-1} \zeta_{k}(a) \cup \zeta_{n-k}(a) \tag{1.5}
\end{equation*}
$$

In fact, we give in (6.17) a formula for the differential of an arbitrary product of the form $\zeta_{i_{1}}\left(a_{1}\right) \cdots \zeta_{i_{n}}\left(a_{n}\right)$ with $a_{k} \in Z^{1}(A)$ and $i_{k} \geq 0$.
1.5. Free binomial graded algebras. A basic example of binomial cup-one algebras is provided by the free binomial graded algebras constructed in section 7. Let $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ be the ring of integrally-valued polynomials with variables from $X$ and with rational coefficients. This is a binomial ring, freely generated as a $\mathbb{Z}$-module by products of polynomials of the form $\binom{x}{n}$, with $x \in X$ and $n \in \mathbb{N}$. We define the free binomial graded algebra on $X$, denoted $\mathrm{T}(X)$, as the tensor algebra on the ideal of all polynomials in $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ without constant term.

We develop here some of the basic properties of these algebras. For instance, we show in Theorem 7.2 that $(\mathrm{T}(X), d)$, with differential $d$ defined by $d(x)=0$ for all $x \in X$, together with the $\cup_{1}-d$ formula and the graded Leibniz rule, is a binomial $\cup_{1}$-dga. Moreover, if $A$ is an any binomial $\cup_{1}$-dga, we show in Corollary 7.6 that there is a bijection between binomial $\cup_{1}$-dga maps $\mathrm{T}(X) \rightarrow A$ and set maps $X \rightarrow Z^{1}(A)$.

Minimal models are then instances of binomial $\cup_{1}$-dga algebras that are free as graded algebras with cup-one products. This definition captures the properties of cup and cupone products that we will use in [27] to construct 1-minimal models over the integers.
1.6. Binomial cup-one dgas over $\mathbb{Z}_{p}$. In section 8 we extend the results in the previous two sections to binomial rings and $\cup_{1}$-dgas over $\mathbb{Z}_{p}$, the prime field of characteristic $p>0$. We say that a commutative $\mathbb{Z}_{p}$-algebra $A$ is a $\mathbb{Z}_{p}$-binomial algebra if $a(a-$ 1) $\cdots(a-n+1)=0$ for all integers $n \geq p$ and all $a \in A$. The binomial operations $\zeta_{n}(a)$ are defined as for algebras over $\mathbb{Z}$, but now only for $1 \leq n \leq p-1$.

With this definition in hand, we proceed as before and introduce the notions of $\mathbb{Z}_{p^{-}}$ binomial $\cup_{1}$-differential graded algebra and free $\mathbb{Z}_{p}$-binomial graded algebra. If $(A, d)$ is a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, we show in Theorem 8.9 that

$$
\begin{equation*}
d \zeta_{n}(a)=-\sum_{k=1}^{n-1} \zeta_{k}(a) \cup \zeta_{n-k}(a) \tag{1.6}
\end{equation*}
$$

for all $a \in A^{1}$ with $d a=0$ and for $2 \leq n \leq p-1$. These notions are further developed in [27] into a theory of 1-minimal models over $\mathbb{Z}_{p}$.
1.7. Ordinary and restricted triple Massey products. We use the constructions above to analyze in some detail special cases of the ordinary Massey products, and also define restricted triple Massey products which have smaller indeterminacy than the usual products. For instance, if $A$ is a binomial $\cup_{1}$-dga over $\mathbb{Z}$, it follows from the existence of binomials, $\zeta_{i}(a)$, satisfying (1.5) that the $n$-fold Massey products $\langle u, \ldots, u\rangle$ are defined and contain 0 , for all $u \in H^{1}(A)$ and $n \geq 3$.

Given a cocycle $a \in C^{1}\left(X ; \mathbb{Z}_{3}\right)$, formula (1.6) is used in the proof of Proposition 9.3 to show that the triple product $\langle[a],[a],[a]\rangle \in H^{2}\left(X ; \mathbb{Z}_{3}\right)$ may be represented by the cocycle $-a \cup \zeta_{2}(a)-\zeta_{2}(a) \cup a$. We show that this triple product contains the negative of the mod 3 Bockstein applied to [a]. More generally, we show in Theorem 9.5 that $p$-fold Massey products $\langle u, \ldots, u\rangle \in H^{2}\left(X ; \mathbb{Z}_{p}\right)$, for odd primes $p$, are defined and contain the negative of the $\bmod p$ Bockstein applied to $u$. Since the indeterminacy of these $p$-fold products in the cohomology of the Eilenberg-MacLane space $K\left(\mathbb{Z}_{p}, 1\right)$ is zero, it follows in general that the $p$-fold product $\langle u, \ldots, u\rangle$ does not necessarily contain zero.

The category of binomial $\cup_{1}$-dgas provides a natural framework for defining generalized Massey products. In section 9 we consider a special case of such cohomology operations, which we call restricted triple Massey products. Furthermore, we relate these products and the binomial operations $\zeta_{n}$ in a binomial $\cup_{1}$-dga.

More precisely, if $A$ is a binomial $\cup_{1}$-dga and $u_{1}, u_{2}$ are elements in $H^{1}(A)$ with $u_{1} \cup$ $u_{2}=0$, we define the restricted Massey product $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$ to be the subset of $H^{2}(A)$
consisting of elements $\left[a_{1} \cup a_{12}-\zeta_{2}\left(a_{1}\right) \cup a_{2}\right.$ ], where each $a_{i}$ is a cocycle with $\left[a_{i}\right]=u_{i}$ and $d a_{12}=a_{1} \cup a_{2}$. We show that these restricted triple Massey products are homotopy invariants with generally smaller indeterminacy than the classical Massey products.

We conclude with a construction of a family of spaces whose homotopy types cannot be distinguished by the usual cup products and triple Massey products, yet can be distinguished using our restricted Massey products. The theory of generalized Massey products continues the program initiated in [26] and is developed more fully in [28, 29], along with further applications.

## 2. Differential graded algebras

We start with some basic definitions. Throughout this section we work over a fixed coefficient ring $R$, assumed to be commutative and with unit $1=1_{R}$.
2.1. Graded algebras. For us, an $R$-algebra is an associative, unital algebra $A$ over a ring $R$. That is to say, $A$ is a ring (with multiplicative identity $1_{A}$ ) which is also an $R$-module, such that the ring and module additions coincide, and scalar multiplication satisfies $r(a b)=(r a) b=a(r b)$ for all $r \in R$ and $a, b \in A$. Consequently, the ring multiplication map, $A \times A \rightarrow A$, is $R$-bilinear, and thus factors through a map $A \otimes_{R} A \rightarrow A$. Moreover, $A$ comes endowed with a structure map, $R \rightarrow A$, given by $r \mapsto r \cdot 1_{A}$. We will assume throughout that the structure map is injective, so that $R$ may be viewed as a subring of $A$.
A graded algebra over $R$ is an $R$-algebra $A$ such that the underlying $R$-module is a direct sum of $R$-modules, $A=\bigoplus_{i \geq 0} A^{i}$, and such that the multiplication map $A \otimes_{R} A \rightarrow A$ sends $A^{i} \otimes_{R} A^{j}$ to $A^{i+j}$.

Observe that $A^{0}$ is a subring of $A$; thus, $1_{A^{0}}$ coincides with $1_{A}$, and so $1=1_{A}$ is a homogeneous element of degree 0 . Consequently, $R$ may be viewed as a subring of $A^{0}$, and the graded pieces $A^{i}$ may be viewed as $A^{0}$-modules. We say that $A$ is connected if the structure map $R \rightarrow A$ maps $R$ isomorphically to $A^{0}$.
We refer to the multiplication maps $\cup: A^{i} \otimes_{R} A^{j} \rightarrow A^{i+j}$, given by $\cup(a \otimes b)=a b$ as the cup-product maps. We may refer to the elements of $A^{i}$ as $i$-cochains, and, when there are other products involved, we will sometimes write $a \cup b$ for the product $a b$. We say that $A$ is graded commutative (for short, $A$ is a cga) if $a b=(-1)^{|a| b \mid} b a$ for all homogeneous elements $a, b \in A$ where $|a|,|b|$ denotes the degrees of $a$ and $b$; respectively. Some basic examples to keep in mind are:
(1) The exterior algebra $\wedge V$, where $V$ is a free $R$-module with generators in odd degrees.
(2) The symmetric algebra $\operatorname{Sym}(V)$, where $V$ is a free $R$-module with generators in even degrees.
(3) The tensor algebra $T(V)$, where $V$ is a free $R$-module of rank at least 2 .

All three examples are graded algebras; the first two are graded-commutative, while the third one is not.

A morphism of graded algebras is a map $\varphi: A \rightarrow B$ of $R$-algebras preserving degrees, that is, $\varphi\left(A^{i}\right) \subset B^{i}$ for all $i$. The set of all such morphisms is denoted by $\operatorname{Hom}(A, B)$.
2.2. Differential graded algebras. A differential graded algebra over a ring $R$ (for short, a dga) is a graded $R$-algebra $A=\bigoplus_{i \geq 0} A^{i}$ endowed with a degree 1 map, $d: A \rightarrow A$, such that $d^{2}=0$ and the following "graded Leibniz rule" is satisfied,

$$
\begin{equation*}
d(a \cup b)=d a \cup b+(-1)^{|a|} a \cup d b \tag{2.1}
\end{equation*}
$$

for all homogenous elements $a, b \in A$.
We denote by $[a] \in H^{i}(A)$ the cohomology class of a cocycle $a \in Z^{i}(A)$. The graded $R$-module $H^{*}(A)$ inherits from $A$ the structure of a graded algebra, with multiplication given by $[a] \cdot[b]=[a b]$. We say that a dga $A$ is graded commutative (for short, $A$ is a cdga) if the underlying graded algebra is a cga. Clearly, if $A$ is a cdga, the $H^{*}(A)$ is a cga.
A morphism of differential graded $R$-algebras is an $R$-linear map, $\varphi: A \rightarrow B$, between two dgas which preserves the grading and commutes with the respective differentials and products; we denote the set of such morphisms by $\operatorname{Hom}(A, B)$. The induced map in cohomology, $\varphi^{*}: H^{*}(A) \rightarrow H^{*}(B)$, is a morphism of graded $R$-algebras. The map $\varphi$ is called a $q$-quasi-isomorphism (for some $q \geq 1$ ) if the induced homomorphism, $\varphi^{*}: H^{i}(A) \rightarrow H^{i}(B)$, is an isomorphism for $i \leq q$ and a monomorphism for $i=q+1$. Two dgas are called $q$-equivalent if there is a zig-zag of $q$-quasi-isomorphisms connecting one to the other. A dga $(A, d)$ is said to be $q$-formal if it is $q$-equivalent to its cohomology algebra, $H^{*}(A)$, endowed with the 0 differential.
2.3. Massey products. A well-known obstruction to formality is provided by the higherorder Massey products, introduced by W.S. Massey in [23, 24]. For simplicity, we focus here on Massey triple products of cohomology classes in degree 1, following the approach from [11, 25].

Let $(A, d)$ be an $R$-dga, and let $u_{1}, u_{2}, u_{3} \in H^{1}(A)$. In general, the Massey triple product $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ is defined if $u_{1} u_{2}=u_{2} u_{3}=0$. If the product is defined, then there are cochains $a_{1}, a_{2}, a_{3}, a_{1,2}, a_{2,3} \in A^{1}$ with the properties that the $a_{i}$ are cocycles, $\left[a_{i}\right]=u_{i}$, and $d a_{i, j}=$ $a_{i} a_{j}$. It follows that $a_{1} a_{2,3}+a_{1,2} a_{3}$ is a 2 -cocycle and $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ then denotes the subset of $H^{2}(A)$ consisting of the cohomology classes $\left[a_{1} a_{2,3}+a_{1,2} a_{3}\right.$ ] for some choice of the cochains $a_{i}, a_{i, j}$. Due to the ambiguity in the choice of the cocycles $a_{i}$, and cochains $a_{i, j}$
the Massey triple product $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ may be viewed as a coset in $H^{2}(A) /\left(u_{1} \cup H^{1}(A)+\right.$ $\left.H^{1}(A) \cup u_{3}\right)$. A Massey product is said to vanish if it is defined and contains the element $0 \in H^{2}(A)$.

If $\varphi: A \rightarrow B$ is a dga morphism, and $\varphi^{*}: H^{*}(A) \rightarrow H^{*}(B)$ is the induced morphism in cohomology, then

$$
\begin{equation*}
\varphi^{*}\left(\left\langle u_{1}, u_{2}, u_{3}\right\rangle\right) \subseteq\left\langle\varphi^{*}\left(u_{1}\right), \varphi^{*}\left(u_{2}\right), \varphi^{*}\left(u_{3}\right)\right\rangle \tag{2.2}
\end{equation*}
$$

In general, this inclusion is strict, but, if $\varphi$ is a 1-quasi-isomorphism, then (2.2) holds as equality. Now, if $A$ is a dga with differential $d: A^{1} \rightarrow A^{2}$ equal to zero, then all Massey triple products $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ vanish. Consequently, Massey triple products are an obstruction to 1 -formality. More precisely, if a dga $(A, d)$ is 1 -formal, and a triple product $\left\langle u_{1}, u_{2}, u_{3}\right\rangle \subset H^{2}(A)$ can be defined, then it must contain the element $0 \in H^{2}(A)$.

These notions extend to higher-order Massey products (in degree one). Suppose $u_{1}, \ldots, u_{n}$ ( $n \geq 3$ ) are elements in $H^{1}(A)$. The $n$-fold product $\left\langle u_{1}, \ldots, u_{n}\right\rangle$ is then defined to be the set of elements in $H^{2}(A)$ represented by cocycles of the form $a_{1} a_{2, n}+a_{1,2} a_{3, n}+\cdots+a_{1, n-1} a_{n}$, where the $a_{i}$ are cocycle representatives of the $u_{i}$, and for $i<j$ the $a_{i, j}$ are cochains in $A^{1}$ which satisfy $d a_{i, j}=a_{i} a_{i+1, j}+a_{i, i+1} a_{i+2, j}+\cdots+a_{i, j-1} a_{j}$ for $1 \leq i<j \leq n$ and $(i, j) \neq(1, n)$.
In subsequent work [29], we develop a theory within which generalized Massey triple and higher order products lead to invariants of 1-equivalence for dgas, and apply this to homotopy classification problems for spaces. Example 9.14 illustrates a particular instance of this approach.

## 3. Steenrod $\cup_{i}$-Products

In this section, we review the properties of Steenrod $\cup_{i}$ products and then focus on properties of the $\cup_{1}$ product of elements in dimension 1 and 2 to define cup-one differential graded algebras.
3.1. The $\cup_{i}$ products. We now enrich the notion of dga with some extra structure. We start with a definition due to Steenrod [33], as developed in subsequent work of Hirsch [17], Kadeishvili [18, 19], Saneblidze [32], and Franz [12, 13].

Definition 3.1. A dga with Steenrod products consists of an $R$-dga ( $A, \cup, d$ ) endowed with $R$-linear maps, $\cup_{i}: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q-i}$, which coincide with the usual cup product when $i=0$, vanish if $p<i$ or $q<i$, and satisfy the identities
(3.2) $(a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+(-1)^{|b|(c \mid-1)}\left(a \cup_{1} c\right) \cup b$
for all homogeneous elements $a, b, c \in A$.

We shall refer to (3.1) as the "Steenrod identities" and to (3.2), with the cup product to the left of the cup-one product, as the "left Hirsch identity".

Lemma 3.2. If $A$ is a dga with Steenrod products, then its cohomology algebra, $H^{*}(A)$, is graded-commutative.

Proof. Let $a \in Z^{p}(A)$ and $b \in Z^{q}(A)$ be two cocycles. Then, by (3.1), $d\left(a \cup_{1} b\right)=$ $(-1)^{p+q}\left(a b+(-1)^{p q} b a\right)$. Hence, $[a][b]+(-1)^{p q}[b][a]=0$, and the claim is proved.

For $a, b, c \in A^{1}$, the identities (3.1) and (3.2) become

$$
\begin{align*}
& d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b,  \tag{3.3}\\
& (a \cup b) \cup_{1} c=a \cup\left(b \cup_{1} c\right)+\left(a \cup_{1} c\right) \cup b . \tag{3.4}
\end{align*}
$$

In particular, if $a, b \in Z^{1}(A)$ are 1-cocyles, then

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-(a \cup b+b \cup a) \tag{3.5}
\end{equation*}
$$

Thus, the operation $\cup_{1}: Z^{1}(A) \otimes_{R} Z^{1}(A) \rightarrow A^{2}$ provides an explicit witness for the noncommutativity of the multiplication map $\cup: Z^{1}(A) \otimes_{R} Z^{1}(A) \rightarrow Z^{2}(A)$ and shows that $u v=-v u$ for elements $u, v \in H^{1}(A)$.

Remark 3.3. In general, the $\cup_{i}$-operations are not associative, even for $i=1$, see for instance $[18,19,22]$. Nevertheless, as we shall see in sections $4-5$, the operation $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ is both associative and commutative for our motivating examples.
3.2. Graded algebras with cup-one products. Let $A=\bigoplus_{p \geq 0} A^{p}$ be a graded algebra over a commutative ring $R$. Recall that we are assuming that the structure map, $R \rightarrow A$, which sends $1_{R}$ to $1_{A}$, is injective, so that $R$ as a subring of $A^{0}$. In what follows, we let $D^{2}(A)$ denote the decomposables in $A^{2}$; that is, the $R$-submodule of $A^{2}$ spanned by all elements of the form $a \cup b$, with $a, b \in A^{1}$.

Definition 3.4. A graded algebra with cup-one products is a graded $R$-algebra $A$ with cup-one product maps, $\cup_{1}: A^{p} \otimes_{R} A^{1} \rightarrow A^{p}$ for $p=1,2$, such that
(i) The left Hirsch identity (3.4) is satisfied for all $a, b, c \in A^{1}$.
(ii) The cup-one map $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ gives the $R$-submodule $R \oplus A^{1} \subset A^{0} \oplus A^{1}$ the structure of a commutative ring.

In (ii), the $R$-module $R \oplus A^{1}$ already possesses three partial multiplication maps, given by the product in the ring $R$, viewed as a map $R \otimes_{R} R \rightarrow R$, and the cup product maps $A^{1} \otimes_{R} R \rightarrow A^{1}$ and $R \otimes_{R} A^{1} \rightarrow A^{1}$, which make $A^{1}$ into an $R$-bimodule. The added structure provided in Definition 3.4 is the cup-one product map, $\cup_{1}: A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$, which meshes with the aforementioned maps to give a multiplication map, $\left(R \oplus A^{1}\right) \otimes_{R}\left(R \oplus A^{1}\right) \rightarrow R \oplus A^{1}$.

Let us highlight the fact that axiom (ii) requires that the cup-one product $\mathrm{U}_{1}: A^{1} \otimes_{R} A^{1} \rightarrow$ $A^{1}$ be both associative and commutative.
Note that, if $A$ is a connected graded algebra with cup-one products, then $A^{\leq 1}=A^{0} \oplus A^{1}$ acquires the structure of a commutative ring. As can be seen in Example 4.2, though, this is not necessarily the case if $A$ is not connected.
For a graded algebra with cup-one products, $A$, the left Hirsch identity expresses the cupone product $D^{2}(A) \otimes_{R} A^{1} \rightarrow A^{2}$ in terms of cup-one products from $A^{1} \otimes_{R} A^{1}$ to $A^{1}$. This raises the question of whether there is an analogous formula that expresses $a \cup_{1}(b \cup c)$ in terms of cup-one products from $A^{1} \otimes_{R} A^{1}$ to $A^{1}$.
Hirsch, [17], in the context of cochain algebras, and Abbassi, [1, Remarque 4.5 (1)], in the context of non-commutative differential forms, give examples to show that there is no general identity relating $a \cup_{1}(b \cup c), a \cup_{1} b$, and $a \cup_{1} c$; see Proposition 4.5 and Example 5.4 below. Nevertheless, these observations do not preclude the possibility of a formula for $a \cup_{1}(b \cup c)$ in terms of cup-one products from $A^{1} \otimes_{R} A^{1}$ to $A^{1}$. We will give in Equation (3.8) such a formula, that we call the right Hirsch identity, in the case when the differential of $a$ is decomposable.
Example 3.5. There is a trivial way to impose a cup-one structure on a dga $(A, d)$ : simply declare all the maps $\cup_{1}: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q-1}$ to be the zero maps.
3.3. Cup-one differential graded algebras. The goal for the rest of this section is to give a formula for the differential of 1-cochains that involves only cup products and cupone products of 1 -cochains. This leads to the following definition, which will play an important role in the sequel.
Definition 3.6. A differential graded $R$-algebra $(A, d)$ is called a $\cup_{1}$-differential graded algebra (for short, $\cup_{1}$-dga) if the following conditions hold.
(i) $A$ is a graded $R$-algebra with cup-one products.
(ii) There is an $R$-linear map $\circ: D^{2}(A) \otimes_{R} D^{2}(A) \rightarrow D^{2}(A)$ such that

$$
\begin{equation*}
(u \cup v) \circ(w \cup z)=\left(u \cup_{1} w\right) \cup\left(v \cup_{1} z\right) \tag{3.6}
\end{equation*}
$$

for all $u, v, w, z \in A^{1}$.
(iii) The differential $d$ satisfies the " $\cup_{1}-d$ formula,"

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b+d b \cup_{1} a-d a \circ d b, \tag{3.7}
\end{equation*}
$$ for all $a, b \in A^{1}$ with $d a, d b \in D^{2}(A)$.

Example 3.7. Let $(A, d)$ be a graded-commutative dga. Put on $A$ the trivial cup-one structure from Example 3.5 and set the o product equal to the zero map. It is readily verified that both the $\cup_{1-d}$ formula and the Steenrod identities are satisfied in this case, and so $(A, d)$ is a $\cup_{1}$-dga.

The next lemma gives a condition for the $\cup_{1}-d$ formula to hold in a dga with Steenrod products.

Lemma 3.8. Let $(A, d)$ be dga such that $A$ is a graded algebra with cup-one products which is endowed with a map $\circ: D^{2}(A) \otimes_{R} D^{2}(A) \rightarrow D^{2}(A)$ such that formula (3.6) holds. Suppose the following two conditions are also satisfied.
(1) The right Hirsch identity is satisfied in degree 1; that is, for all a, $b, c \in A^{1}$ with $d a \in D^{2}(A)$,

$$
\begin{equation*}
a \cup_{1}(b \cup c)=d a \circ(b \cup c)-(b \cup c) \cup_{1} a . \tag{3.8}
\end{equation*}
$$

(2) The Steenrod identity is satisfied in degree 1; that is, for all $a, b \in A^{1}$,

$$
\begin{equation*}
d\left(a \cup_{1} b\right)=-a \cup b-b \cup a+d a \cup_{1} b-a \cup_{1} d b . \tag{3.9}
\end{equation*}
$$

Then the $\cup_{1}-d$ formula (3.7) holds, and so $(A, d)$ is a $\cup_{1}-d g a$. In particular, if $A$ is a dga with Steenrod products and equations and (3.6) and (3.8) are satisfied, then $A$ is a $\cup_{1}-d g a$.

Proof. Let $a, b \in A^{1}$ with $d a, d b \in D^{2}(A)$. From the assumptions that $d a$ and $d b$ are decomposable it follows from equation (3.8) and the linearity of the $U_{1}$ and $\circ$ products that

$$
\begin{equation*}
a \cup_{1} d b=(d a) \circ(d b)-d b \cup_{1} a . \tag{3.10}
\end{equation*}
$$

Combining this equation with (3.9), the claim follows.
Question 3.9. Suppose $(A, d)$ is a $\cup_{1}$-dga. Does the cohomology algebra $H^{*}(A)$, with cup-product inherited from $A$ and with zero differential, also inherit in a natural way a $\cup_{1}$-product from $A$ that makes it into a graded algebra with cup-one products?
3.4. Functoriality. A morphism of $\cup_{1}$-dgas is a dga map $\varphi: A \rightarrow B$ between two such objects which commutes with the cup-one products, that is, $\varphi\left(a_{1} \cup_{1} a_{2}\right)=\varphi\left(a_{1}\right) \cup_{1} \varphi\left(a_{2}\right)$, for all $a_{1}, a_{2} \in A^{1}$. We denote the set of all such morphisms by $\operatorname{Hom}_{1}(A, B)$. Clearly, we have an inclusion $\operatorname{Hom}_{1}(A, B) \subseteq \operatorname{Hom}(A, B)$, but in general this inclusion is strict, as illustrated in the following example.

Example 3.10. Let $A=\wedge(a)$ be the exterior algebra over $R=\mathbb{Z}$ on a single generator $a$ in degree 1 , with $d a=0$ and with $a \cup_{1} a=a$. Then $\operatorname{Hom}(A, A) \cong \mathbb{Z}$ while $\operatorname{Hom}_{1}(A, A)$ contains only two elements, namely, the zero map and the identity.

The notions of 1-quasi-isomorphism and 1-equivalence are defined in the category of $\cup_{1-}{ }^{-}$ dgas exactly as in the category of dgas. Note though that two $\cup_{1}$-dgas can be 1-equivalent as dgas, but not as $\cup_{1}$-dgas. For instance, take $A$ to be the $\cup_{1}$-dga from Example 3.10, and $A^{\prime}$ to be the same dga, but with trivial cup-one structure, $a \cup_{1} a=0$.

It would be interesting to have a definition of 1-formality in this context, but this depends on answering Question 3.9 first. If this could be done, it would also be interesting to decide whether it is possible that a $\cup_{1}$-dga may be 1-formal as a dga, but not as a $\cup_{1}$-dga.

## 4. Cochain algebras

The first of our two motivating examples for the definition of cup-one dgas is the cochain complex of a space. In this section we review the notion of a $\Delta$-complex and the cochain complex of a $\Delta$-set. Theorem 4.1 relates the $n$-type of CW complexes to equivalences of cochain complexes.
4.1. $\Delta$-sets and $\Delta$-complexes. We start the section by reviewing the notion of a $\Delta$ complex, in the sense of Rourke and Sanderson [30]; see also Hatcher [16] and Friedman [14]. We will view such a complex as the geometric realization of the corresponding $\Delta$ set, cf. [14].

An (abstract) $n$-simplex $\Delta^{n}$ is simply a finite ordered set, $(0,1, \ldots, n)$. The face maps $d_{i}: \Delta^{n} \rightarrow \Delta^{n-1}$ for $n \geq 1$ are given by omitting the $i$-th element in the set; that is,

$$
\begin{equation*}
d_{i}(0, \ldots, n)=(0, \ldots, \hat{i}, \ldots, n) \tag{4.1}
\end{equation*}
$$

These maps satisfy $d_{i} d_{j}=d_{j-1} d_{i}$ whenever $0 \leq i<j \leq n$ and $n \geq 2$. The geometric realization of the simplex, $\left|\Delta^{n}\right|$, is the convex hull of $n+1$ affinely independent vectors in $\mathbb{R}^{n+1}$, endowed with the subspace topology; the face maps induce continuous maps, $d_{i}:\left|\Delta^{n}\right| \rightarrow|\Delta|^{n-1}$.
More generally, a $\Delta$-set consists of a sequence of sets $\left\{X_{n}\right\}_{n \geq 0}$ and maps $d_{i}: X_{n+1} \rightarrow X_{n}$ for each $0 \leq i \leq n+1$ such that $d_{i} d_{j}=d_{j-1} d_{i}$ whenever $i<j$. This is the generalization of the notion of ordered (abstract) simplicial complex, where the sets $X_{n}$ are the sets of $n$-simplices and the maps $d_{i}$ are the face maps. Note that the singular simplices in a topological space form a $\Delta$-set.
The geometric realization of a $\Delta$-set is the topological space

$$
\begin{equation*}
|X|=\coprod_{n \geq 0} X_{n} \times\left|\Delta^{n}\right| / \sim, \tag{4.2}
\end{equation*}
$$

where $\sim$ is the equivalence relation generated by $\left(x, \partial_{i}(p)\right) \sim\left(d_{i}(x), p\right)$ for $x \in X_{n+1}$, $p \in\left|\Delta^{n}\right|$, and $0 \leq i \leq n$. Such a space is called a $\Delta$-complex, and can be viewed either as a special kind of CW-complex, or a generalized simplicial complex.
The assignment $X \leadsto|X|$ is functorial: if $f: X \rightarrow Y$ is a map of $\Delta$-sets (i.e., $f$ is a family of maps $f_{n}: X_{n} \rightarrow Y_{n}$ commuting with the face maps), there is an obvious realization,
$|f|:|X| \rightarrow|Y|$, and this is a (continuous) map of $\Delta$-complexes. We will often abuse notation and mistake an (abstract) $\Delta$-complex for its geometric realization, $|X|$, and likewise for maps between these objects.
4.2. Cochain algebras of $\Delta$-complexes. The chain complex of a $\Delta$-set $X$, denoted $C_{*}(X)$, coincides with the simplicial chain complex of its geometric realization: for each $n \geq$ 0 , the chain group $C_{n}(X)$ is the free abelian group on $X_{n}$, while the boundary maps $\partial_{n}: C_{n}(X) \rightarrow C_{n-1}(X)$ are the $\mathbb{Z}$-linear maps given on basis elements by $\partial_{n}=\sum_{i=0}^{n}(-1)^{i} d_{i}$, where $d_{i}$ is given by formula (4.1). The cochain complex $C^{*}(X)=\left(C^{n}(X), \delta^{n}\right)_{n \geq 0}$ is defined similarly with $(\delta u)(x)=u(\partial x)$ for $u \in C^{n}(X)$ and $x \in C_{n+1}(X)$.
More generally, we let $C=C^{*}(X ; R)$ be the cochain complex of $X$ with coefficients in a commutative ring $R$ with unit 1. This $R$-module acquires the structure of an $R$-algebra, with multiplication given by the cup-product of cochains. More precisely, if $u \in C^{p}(X ; R)$ and $v \in C^{q}(X ; R)$, then $u \cup v \in C^{p+q}(X ; R)$ is the cochain given by

$$
\begin{equation*}
u \cup v([0,1, \ldots, p+q])=u([0, \ldots, p]) \cdot v([p, \ldots, p+q]) \tag{4.3}
\end{equation*}
$$

where $\left[a_{0}, a_{1}, \ldots, a_{k}\right]$ denotes a $k$-simplex on the indicated vertices and $\cdot$ denotes the product in $R$. Moreover, the unit $1_{C} \in C^{0}(X ; R)$ is the 0 -cochain which takes the value $1_{R}$ on each 0 -simplex; clearly, the structure map $R \rightarrow C$ which takes $1_{R}$ to $1_{C}$ is injective.

Two maps of spaces, $f, g: X \rightarrow Y$, are said to be $n$-homotopic (in the sense of [35]), if $f \circ h \simeq g \circ h$, for every map $h: K \rightarrow X$ from a CW-complex $K$ of dimension at most $n$. A map $f: X \rightarrow Y$ is an $n$-homotopy equivalence (for some $n \geq 1$ ) if it admits an $n$-homotopy inverse. If such a map $f$ exists, one says that $X$ and $Y$ have the same n-homotopy type, written $X \simeq_{n} Y$. Two CW-complexes, $X$ and $Y$, are said to be of the same n-type if $X^{(n)} \simeq_{n-1} Y^{(n)}$. In particular, any two connected CW-complexes have the same 1-type, and they have the same 2-type if and only if their fundamental groups are isomorphic.

Continuous maps between CW-complexes (or $\Delta$-complexes, or simplicial complexes) can be approximated up to homotopy by maps that respect cellular structures; we will do that when needed, without further mention. A map $f: X \rightarrow Y$ between two $\Delta$-complexes induces a dga map, $f^{\sharp}: C^{*}(Y ; R) \rightarrow C^{*}(X ; R)$, between the respective cochain algebras, and thus a morphism, $f^{*}: H^{*}(Y ; R) \rightarrow H^{*}(X ; R)$, between their cohomology algebras. If $f$ is a homotopy equivalence, then $f^{\sharp}$ is a quasi-isomorphism of $R$-dgas. More generally, we have the following theorem, which provides the bridge from topology to algebra in this context. Although surely known to specialists, we could not find a proof in the literature, so we include one here.

Theorem 4.1. If $X$ and $Y$ are $C W$-complexes of the same $n$-type, then the cochain algebras $C^{*}(X ; R)$ and $C^{*}(Y ; R)$ are $(n-1)$-equivalent.

Proof. By assumption, $X^{(n)} \simeq_{n-1} Y^{(n)}$. Hence, by [35, Theorem 6], there is a homotopy equivalence, $f$, from $\bar{X}^{(n)}=X^{(n)} \vee \bigvee_{i \in I} S_{i}^{n}$ to $\bar{Y}^{(n)}=Y^{(n)} \vee \bigvee_{j \in J} S_{j}^{n}$, for some indexing sets $I$ and $J$. Let $q_{X}: \bar{X}^{(n)} \rightarrow X^{(n)}$ and $q_{Y}: \bar{Y}^{(n)} \rightarrow Y^{(n)}$ be the maps that collapse the wedges of $n$-spheres to the basepoint of the wedge.


Clearly, the map $f^{\sharp}: C^{*}\left(\bar{Y}^{(n)} ; R\right) \rightarrow C^{*}\left(\bar{X}^{(n)} ; R\right)$ is a quasi-isomorphism, whereas the maps $q_{X}^{\sharp}: C^{*}\left(X^{(n)} ; R\right) \rightarrow C^{*}\left(\bar{X}^{(n)} ; R\right)$ and $q_{Y}^{\sharp}: C^{*}\left(Y^{(n)} ; R\right) \rightarrow C^{*}\left(\bar{Y}^{(n)} ; R\right)$ are (n-1)-quasiisomorphisms. Therefore, $C^{*}\left(X^{(n)} ; R\right)$ and $C^{*}\left(Y^{(n)} ; R\right)$ are $(n-1)$-equivalent. Hence, $C^{*}(X ; R)$ and $C^{*}(Y ; R)$ are also ( $n-1$ )-equivalent.

In particular, if $X$ and $Y$ are connected CW-complexes of the same 2-type, i.e., if $\pi_{1}(X) \cong$ $\pi_{1}(Y)$, then $C^{*}(X ; R)$ is 1-equivalent to $C^{*}(Y ; R)$.
4.3. The Steenrod $\cup_{i}$ operations on cochains. Let $X$ be a $\Delta$-complex, and let $A=$ $\left(C^{*}(X ; R), \cup, \delta\right)$ be its cochain algebra with coefficients in a commutative ring $R$. In a seminal paper from 1947, Steenrod [33] introduced operations $\cup_{i}: A^{p} \otimes_{R} A^{q} \rightarrow A^{p+q-i}$ that now bear his name. For $i=0$, the $\cup_{0}$ operation coincides with the usual cup product, while if $p<i$ or $q<i$, then $\cup_{i}=0$. Crucially, Steenrod's $\cup_{i}$ products satisfy the identities (3.1). Several years later, Hirsch [17] showed that the identities (3.2) also hold. In our terminology from Definition 3.1, the cochain algebra $A$ is a dga with Steenrod products.
The construction of the $\cup_{i}$ products is functorial, in the following sense. Let $f: X \rightarrow Y$ be a map of $\Delta$-complexes. Without loss of generality, we may take baycentric subdivisons on $X$ and $Y$. Now put partial orders on the resulting sets of vertices by assigning to each vertex the dimension of the simplex for which it is the barycenter of; note that this defines a linear order on each simplex of $X$ and $Y$. With these constructions, the map $f$ induces a chain map $f^{\sharp}: C^{*}(Y) \rightarrow C^{*}(X)$ that preserves the ordering of the vertices. Thus, by [33, Theorem 3.1], the map $f^{\sharp}$ also commutes with the $\cup$ and $\cup_{i}$ products.
The following is the definition from [33] of the $\cup_{1}$-products of cochains $u \in A^{p}, v \in A^{q}$, when evaluated on a simplex,

$$
\begin{gather*}
u \cup_{1} v([0,1, \ldots, p+q-1])=\sum_{j=0}^{p-1}(-1)^{(p-j)(q+1)} u([0, \ldots, j, j+q, \ldots, p+q-1])  \tag{4.5}\\
v([j, \ldots, j+q])
\end{gather*}
$$

Clearly, $u \cup_{1} v=0$ if either $u$ or $v$ is a 0 -cochain. When both $u$ and $v$ are 1-cochains, formulas (4.3) and (4.5) simplify to

$$
\begin{align*}
(u \cup v)(s) & =u\left(e_{1}\right) \cdot v\left(e_{2}\right),  \tag{4.6}\\
\left(u \cup_{1} v\right)(e) & =u(e) \cdot v(e), \tag{4.7}
\end{align*}
$$

where in (4.6), $s=[i, j, k]$ is a 2 -simplex with front face $e_{1}=[i, j]$ and back face $e_{2}=[j, k]$, while in (4.7), $e$ is a 1 -simplex. In particular, the $\cup_{1}$-product on $C^{1}(X ; R)$ is both associative and commutative. Therefore, the $R$-module $R \oplus C^{1}(X ; R)$ naturally acquires the structure of a commutative ring with unit. Since, as mentioned previously, the left Hirsch identity (3.4) holds, the cochain algebra $C^{*}(X ; R)$ is a graded algebra with cup-one products, in the sense of Definition 3.4.
Example 4.2. Let $I$ be the closed interval [ 0,1 ], viewed as a simplicial complex in the usual way, and let $C=C^{*}(I ; R)$ be its cochain algebra over $R$. Then $C^{0}=R \oplus R$ with generators $t_{0}, t_{1}$ corresponding to the endpoints 0 and 1 , and $C^{1}=R$ with generator $u$. The differential $d: C^{0} \rightarrow C^{1}$ is given by $d t_{0}=-u$ and $d t_{1}=u$, while the multiplication is given on generators by $t_{i} t_{j}=\delta_{i j} t_{i}, t_{0} u=u t_{1}=u$, and $u t_{0}=t_{1} u=0$. Note that the cocycle $t_{0}+t_{1}$ is the unit of $C$, and that multiplication of 0 - and 1-cocycles is not commutative, e.g., $t_{0} u \neq u t_{0}$. Furthermore, $H^{*}(C)=R$, concentrated in degree 0 . Finally, the cup-one product $C^{1} \otimes_{R} C^{1} \rightarrow C^{1}$ is given by $u \cup_{1} u=u$.
Example 4.3. Let $X=S^{1} \times S^{1}$ be the 2-torus, and consider the $\Delta$-complex structure on $X$ depicted in Figure 1. The attaching maps of the standard 2 -simplex are indicated by the numbers on the vertices and the requirement that the correspondence preserves the ordering. Edges are oriented from the lower-numbered vertex to higher-numbered vertex. The 1 -cochains are represented by 1 -cells-with a transverse orientation-in a graph whose edges are transverse to the edges of the $\Delta$-complex with vertices of the graph contained in the 2 -cells of the $\Delta$-complex. The value of a 1 -cochain, $u$, on an oriented 1 -cell, $e$, is 0 if $u$ and $e$ do not intersect. If $u$ and $e$ intersect, then $u(e)=1$ if at the point of intersection the transverse orientation of $u$ agrees with the orientation of $e$; otherwise, $u(e)=-1$. Now write $\pi_{1}(X)=\mathbb{Z}^{2}=\left\langle x_{1}, x_{2} \mid x_{1} x_{2} x_{1}^{-1} x_{2}^{-1}\right\rangle$, and identify $H^{*}(X)$ with $\bigwedge^{*}\left(v_{1}, v_{2}\right)$, the exterior algebra on two generators in degree 1 . Let $a_{1}$ and $a_{2}$ be the 1 -cocycles shown in the figure, so that $\left[a_{i}\right]=v_{i}$, and let $b$ be the indicated 1-cochain. Then $b=a_{1} \cup_{1} a_{2}$ and $d(b)=-a_{1} \cup a_{2}-a_{2} \cup a_{1}$.
4.4. Cochain algebras as cup-one differential graded algebras. We are now ready to state and prove the main result of this section. Let $X$ be a non-empty $\Delta$-complex, and let $R$ be a unital commutative ring.
Theorem 4.4. Let $A=C^{*}(X ; R)$ be a cochain algebra. Then $A$ is a $\cup_{1}-d g a$.
Proof. As mentioned previously, $C^{*}(X ; R)$ is a graded algebra with cup-one products. Furthermore, it is a differential graded algebra, and the Steenrod identities (3.1) hold


Figure 1. Cochains on a torus
in full generality; in particular, (3.9) holds. In view of Lemma 3.8, we only need to show that there is a well-defined $\circ$ map satisfying (3.6) and that the right Hirsch identity, equation (3.8), also holds.
From formula (4.7) it follows that the $\cup_{1}$-product is both associative and commutative for 1-cochains. Therefore, $R \oplus C^{1}(X ; R)$ naturally acquires the structure of a commutative ring with unit, in the fashion outlined right after Definition 3.4. From formulas (4.6)(4.7), it follows that for 1 -cochains $u, v, w, u_{1}$, and $u_{2}$ and for a 2-simplex $s$ in $X$ we have that

$$
\begin{align*}
u\left(e_{1}\right)(v \cup w)(s) & =\left(\left(u \cup_{1} v\right) \cup w\right)(s),  \tag{4.8}\\
u\left(e_{2}\right)(v \cup w)(s) & =\left(v \cup\left(u \cup_{1} w\right)\right)(s),  \tag{4.9}\\
\left(u_{1} \cup u_{2}\right)(s) \cdot(v \cup w)(s) & =\left(\left(u_{1} \cup_{1} v\right) \cup\left(u_{2} \cup_{1} w\right)\right)(s) . \tag{4.10}
\end{align*}
$$

Define now an $R$-linear map $\circ: A^{2} \otimes_{R} A^{2} \rightarrow A^{2}$ by setting

$$
\begin{equation*}
(v \circ w)(s)=v(s) \cdot w(s) \tag{4.11}
\end{equation*}
$$

for any 2-cochains $v, w$ and any 2-simplex $s$. It follows from (4.6), (4.7), and (4.10) that the map above restricts to a map $\circ: D^{2}(A) \otimes_{R} D^{2}(A) \rightarrow D^{2}(A)$ which obeys formula (3.6).
The next step is to show that the right Hirsch identity holds; that is, if $d u=\sum_{i} u_{1, i} \cup u_{2, i}$, then

$$
\begin{equation*}
u \cup_{1}(v \cup w)=-\left(u \cup_{1} v\right) \cup w+\sum_{i}\left(u_{1, i} \cup_{1} v\right) \cup\left(u_{2, i} \cup_{1} w\right)-v \cup\left(u \cup_{1} w\right) . \tag{4.12}
\end{equation*}
$$

By equation (4.5), the cup-one product map $\cup_{1}: A^{1} \otimes_{R} A^{2} \rightarrow A^{2}$ is given by

$$
\begin{equation*}
\left(u \cup_{1} z\right)(s)=-u\left(e_{3}\right) z(s), \tag{4.13}
\end{equation*}
$$

for $u \in A^{1}$ and $z \in A^{2}$. Since $d u=u\left(e_{1}\right)+u\left(e_{2}\right)-u\left(e_{3}\right)$, it follows that

$$
\begin{equation*}
\left(u \cup_{1} z\right)(s)=\left(-u\left(e_{1}\right)-u\left(e_{2}\right)+d u(s)\right) \cdot z(s) \tag{4.14}
\end{equation*}
$$

For $z=v \cup w$ and $d u=\sum_{i} u_{1, i} \cup u_{2, i}$, it then follows from equations (4.8), (4.9), and (4.10) that

$$
\begin{equation*}
\left(u \cup_{1}(v \cup w)\right)(s)=-\left(\left(u \cup_{1} v\right) \cup w-\sum_{i}\left(u_{1, i} \cup_{1} v\right) \cup\left(u_{2, i} \cup_{1} w\right)+v \cup\left(u \cup_{1} w\right)\right)(s) \tag{4.15}
\end{equation*}
$$

Since equation (4.15) holds for all 2-simplices, $s$, the proof is complete.
The next result is motivated by-and generalizes-an example due to Hirsch [17].
Proposition 4.5. There is no linear combination of cup products and cup products of cup-one products that equals $u_{3} \cup_{1}\left(u_{1} \cup u_{2}\right)$ for all 1-cochains $u_{1}, u_{2}, u_{3} \in C^{1}(X ; R)$ and all $\Delta$-complexes $X$.

Proof. We need to show that there are no constants $\alpha\left(i_{1}, i_{2} \mid i_{3}\right), \alpha\left(j_{1} \mid j_{2}, j_{3}\right), \alpha\left(k_{1}, k_{2} \mid k_{3}, k_{4}\right)$, and $\alpha\left(\ell_{i} \mid \ell_{2}\right)$ with $1 \leq i_{r} \leq 3,1 \leq j_{r} \leq 3,1 \leq k_{r} \leq 3$, and $1 \leq \ell_{r} \leq 3$, such that the equation

$$
\begin{align*}
u_{3} \cup_{1}\left(u_{1} \cup u_{2}\right)= & \sum_{1 \leq i_{t} \leq 3} \alpha\left(i_{1}, i_{2} \mid i_{3}\right)\left(u_{i_{1}} \cup_{1} u_{i_{2}}\right) \cup u_{i_{3}}+\sum_{1 \leq j_{t} \leq 3} \alpha\left(j_{1} \mid j_{2}, j_{3}\right) u_{j_{1}} \cup\left(u_{j_{2}} \cup_{1} u_{j_{3}}\right)  \tag{4.16}\\
& +\sum_{1 \leq k_{t} \leq 3} \alpha\left(k_{1}, k_{2} \mid k_{3}, k_{4}\right)\left(u_{k_{1}} \cup_{1} u_{k_{2}}\right) \cup\left(u_{k_{3}} \cup_{1} u_{k_{4}}\right)+\sum_{1 \leq l_{i} \leq 3} \alpha\left(l_{i} \mid l_{j}\right) u_{l_{i}} \cup u_{l_{j}}
\end{align*}
$$

holds for all 1-cochains, $u_{1}, u_{2}, u_{3}$, on an arbitrary $\Delta$-complexes $X$.
Let $X$ be the standard 2-simplex $[0,1,2]$, with front face $e_{1}=[0,1]$, back face $e_{2}=[1,2]$, and third face $e_{3}=[0,2]$. Let $n$ be an integer and define 1-cochains on $[0,1,2]$ by

$$
\begin{aligned}
& u_{1}\left(e_{i}\right)=1 \text { if } i=1 \text { and } 0 \text { otherwise, } \\
& u_{2}\left(e_{i}\right)=1 \text { if } i=2 \text { and } 0 \text { otherwise, } \\
& u_{3}\left(e_{i}\right)=n \text { if } i=3 \text { and } 0 \text { otherwise. }
\end{aligned}
$$

By direct computation of cup and cup-one products of these cochains, it follows that the right hand side of equation (4.16) is equal to

$$
\begin{aligned}
& \alpha(1,1 \mid 2)\left(u_{1} \cup_{1} u_{1}\right) \cup u_{2}+\alpha(1 \mid 2,2) u_{1} \cup\left(u_{2} \cup_{1} u_{2}\right) \\
& \quad+\alpha(1,1 \mid 2,2)\left(u_{1} \cup_{1} u_{1}\right) \cup\left(u_{2} \cup_{1} u_{2}\right)+\alpha(1 \mid 2) u_{1} \cup u_{2} \\
& \quad=\alpha(1,1 \mid 2)+\alpha(1 \mid 2,2)+\alpha(1,1 \mid 2,2)+\alpha(1 \mid 2)
\end{aligned}
$$

From the definition of the cup-one product, we have that

$$
u_{3} \cup_{1}\left(u_{1} \cup u_{2}\right)([0,1,2])=-u_{3}\left(e_{3}\right) \cdot\left(u_{1} \cup u_{2}\right)([0,1,2])=-n .
$$

Thus, assuming that equation (4.16) holds in our situation, we have that

$$
\begin{equation*}
-n=\alpha(1,1 \mid 2)+\alpha(1 \mid 2,2)+\alpha(1,1 \mid 2,2)+\alpha(1 \mid 2) \tag{4.17}
\end{equation*}
$$

for all integers $n$. This is a contradiction, since the right hand side of equation (4.17) is a constant, and so we are done.

## 5. Non-commutative differential forms

The second of our two motivating examples for the definition of cup-one dgas is the algebra of non-commutative differential forms. In this section we review the definition and properties of non-commutative differential forms and show that they are cup-one differential graded algebras.
5.1. The algebra of non-commutative differential forms. We start this section with a brief review of the theory of non-commutative differential forms, as developed by H. Cartan [6], A. Connes [7], and M. Karoubi [20, 21], as well as some more recent developments due to N. Battikh [2, 3] and A. Abbassi [1].

Let $R$ be a commutative ring with unit 1, and let $A$ be an $R$-algebra with injective structure map $R \rightarrow A$. For each $n \geq 0$, let $T^{n}(A)=A \otimes_{R} \cdots \otimes_{R} A$ be the $(n+1)$-fold tensor product of $A$ (over $R$ ).
Proposition 5.1. The $R$-module $T^{*}(A)=\bigoplus_{n \geq 0} T^{n}(A)$ has the structure, of an $R$-dga with multiplication $T^{n} \otimes_{R} T^{m} \rightarrow T^{n+m}$ and differential $D: T^{n} \rightarrow T^{n+1}$ given by

$$
\begin{align*}
& \left(a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n}\right) \cup\left(b_{0} \otimes b_{1} \otimes \cdots \otimes b_{m}\right)=a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n} b_{0} \otimes b_{1} \otimes \cdots \otimes b_{m},  \tag{5.1}\\
& D\left(a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n}\right)=1 \otimes a_{1} \otimes \cdots \otimes a_{n}-a_{0} \otimes 1 \otimes \cdots \otimes a_{n}+\cdots  \tag{5.2}\\
& \quad+(-1)^{n+1} a_{0} \otimes \cdots \otimes a_{n} \otimes 1 .
\end{align*}
$$

In particular, $D(a)=1 \otimes a-a \otimes 1$ and $D\left(a_{0} \otimes a_{1}\right)=1 \otimes a_{0} \otimes a_{1}-a_{0} \otimes 1 \otimes a_{1}+a_{0} \otimes a_{1} \otimes 1$.
Now set $\Omega^{0}(A)=A$ and let $\Omega^{1}(A)$ be the kernel of the multiplication map $A \otimes_{R} A \rightarrow A$. Clearly, the $R$-module $\Omega^{1}(A)$ is also an $A$-bimodule. For each $n \geq 1$, we let

$$
\begin{equation*}
\Omega^{n}(A)=\Omega^{1}(A) \otimes_{A} \cdots \otimes_{A} \Omega^{1}(A) \tag{5.3}
\end{equation*}
$$

be the $n$-fold tensor product of $\Omega^{1}(A)$ over $A$. Since $A \otimes_{A} A=A$, we have that $\Omega^{n}(A) \subset$ $T^{n}(A)$. Thus, the $R$-module $\Omega^{*}(A)=\bigoplus_{n \geq 0} \Omega^{n}(A)$ inherits a natural $R$-dga structure from $T^{*}(A)$; we denote its differential by $d$.
Writing $\bar{A}=A / R$, it is readily seen that the map $A \otimes_{R} \bar{A} \rightarrow \Omega^{1}(A), x \otimes \bar{y} \mapsto x d y$ is an isomorphism. Hence,

$$
\begin{equation*}
\Omega^{n}(A) \cong A \otimes_{R} \bar{A} \otimes_{R} \cdots \otimes_{R} \bar{A}, \tag{5.4}
\end{equation*}
$$

where the number of factors of $\bar{A}$ is $n$. The elements of this $R$-module, known as noncommutative forms of degree $n$, can be viewed as linear combinations of elements of the form $a_{0} d a_{1} \cdots d a_{n}$. The differential $d$ given by

$$
\begin{equation*}
d\left(a_{0} d a_{1} \cdots d a_{n}\right)=d a_{0} d a_{1} \cdots d a_{n} \tag{5.5}
\end{equation*}
$$

The canonical projection, $J: T^{*}(A) \rightarrow \Omega^{*}(A), a_{0} \otimes a_{1} \otimes \cdots \otimes a_{n} \mapsto a_{0} d a_{1} \cdots d a_{n}$ is a morphism of graded $R$-modules which is compatible with the differentials, but not with the algebra structures. Nevertheless, $J(\alpha \beta)=J(\alpha) J(\beta)$ if $\beta$ is a cocycle in $T^{*}(A)$.
5.2. Cup- $i$ products in $T(A)$. In [2, §3], Battikh defines $R$-linear maps $\cup_{i}: T^{n}(A) \otimes_{R}$ $T^{m}(A) \rightarrow T^{n+m-i}(A)$ which vanish if $i>\min (n, m)$, coincide with the multiplication in $T^{*}(A)$ if $i=0$, and satisfy Steenrod's identities,

$$
\begin{equation*}
D\left(\alpha \cup_{i} \beta\right)=(-1)^{n+m-i} \alpha \cup_{i-1} \beta+(-1)^{n m+n+m} \beta \cup_{i-1} \alpha+D \alpha \cup_{i} \beta+(-1)^{n} \alpha \cup_{i} D \beta . \tag{5.6}
\end{equation*}
$$

While these maps are rather difficult to write down explicitly, here is the definition of the $\cup_{1}$ map. Let $\alpha=a_{0} \otimes \cdots \otimes a_{p}$ and $\beta=b_{0} \otimes \cdots \otimes b_{q}$; then ([2, Proposition 3.1]):

$$
\begin{align*}
\left(a_{0} \otimes \cdots \otimes a_{p}\right) \cup_{1}\left(b_{0} \otimes \cdots \otimes b_{q}\right)= & \sum_{i=0}^{p-1}(-1)^{(p-i)(q+1)} a_{0} \otimes \cdots \otimes a_{i-1} \otimes a_{i} b_{0} \otimes  \tag{5.7}\\
& b_{1} \otimes \cdots \otimes b_{q-1} \otimes b_{q} a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{p} .
\end{align*}
$$

In particular,

$$
\begin{equation*}
\left(a_{0} \otimes a_{1}\right) \cup_{1}\left(b_{0} \otimes b_{1}\right)=a_{0} b_{0} \otimes b_{1} a_{1} \tag{5.8}
\end{equation*}
$$

Now suppose $A$ is a commutative $R$-algebra. Then the (left) Hirsch formula holds in all degrees ( $\left[1\right.$, Proposition 4.4]); namely, if $\alpha_{i} \in T^{n_{i}}(A)$, then

$$
\begin{equation*}
\left(\alpha_{1} \cup \alpha_{2}\right) \cup_{1} \alpha_{3}=\alpha_{1} \cup\left(\alpha_{2} \cup_{1} \alpha_{3}\right)+(-1)^{n_{1}\left(n_{2}+1\right)}\left(\alpha_{1} \cup_{1} \alpha_{3}\right) \cup \alpha_{2} . \tag{5.9}
\end{equation*}
$$

Since $T(A)$ satisfies equations (5.6) and (5.9), it is a dga with Steenrod products, in the sense of Definition 3.1. Moreover, formula (5.8) shows that the $\cup_{1}$ product on $T^{1}(A)$ is associative and commutative; therefore, it gives $R \oplus T^{1}(A)$ the structure of a commutative ring with unit. Hence, $T(A)$ is a graded algebra with cup-one products, in the sense of Definition 3.4.
5.3. Cup- $i$ products in $\Omega(A)$. The $\cup_{i}$ operations on $T^{*}(A)$ induce $\cup_{i}$ operations on $\left(\Omega^{*}(A), d\right)$ that satisfy Steenrod's identites, cf. [2, Proposition 3.6]. The $\cup_{1}$ operation on $\Omega^{1}(A)$ takes the explicit form indicated in the next lemma.
Lemma 5.2. Let $a_{0} d a_{1}$ and $b_{0} d b_{1}$ be two elements in $\Omega^{1}(A)$. Then,

$$
\begin{aligned}
a_{0} d a_{1} \cup_{1} b_{0} d b_{1}= & a_{0} d\left(a_{1} b_{0} b_{1}\right)-a_{0} b_{1} d\left(a_{1} b_{0}\right)-a_{0} a_{1} b_{0} d b_{1}-a_{0} a_{1} d\left(b_{0} b_{1}\right) \\
& +a_{0} a_{1} b_{1} d b_{0}+a_{0} a_{1} b_{0} d b_{1} .
\end{aligned}
$$

In particular, $d a \cup_{1} d b=d(a b)-b d a-a d b$, and so $d a \cup_{1} d a=d\left(a^{2}\right)-2 a d a$.
Proof. Recall that $\Omega^{*}=\Omega^{*}(A)$ is an $A$-bimodule. Furthermore, the differential $d: \Omega^{0} \rightarrow$ $\Omega^{1}$ satisfies the product rule, $d(a b)=(d a) \cdot b+a \cdot(d b)$. The first step is to write $d a \cdot b$ as an element in $\Omega^{1}$ in normal form (i.e., as a sum of terms of the form $x_{i} d y_{i}$ ). This can be done using the product rule, or, alternatively, the following computation:

$$
\begin{align*}
d a \cdot b & =(1 \otimes a-a \otimes 1) b=1 \otimes a b-a \otimes b \\
& =1 \otimes a b-a b \otimes 1+a b \otimes 1-a \otimes b  \tag{5.10}\\
& =d(a b)+a(b \otimes 1-1 \otimes b)=d(a b)-a d b .
\end{align*}
$$

Therefore,

$$
\begin{aligned}
d a \cup_{1} d b & =(1 \otimes a-a \otimes 1) \cup_{1}(1 \otimes b-b \otimes 1)=1 \otimes a b-b \otimes a-a \otimes b+a b \otimes 1 \\
& =1 \otimes a b-a b \otimes 1+a b \otimes 1-b \otimes a+b a \otimes 1-a b \otimes 1-a(1 \otimes b-b \otimes 1) \\
& =d(a b)-b(1 \otimes a-a \otimes 1)-a d b=d(a b)-b d a-a d b .
\end{aligned}
$$

The general case follows along the same lines:

$$
\begin{aligned}
a_{0} d a_{1} \cup_{1} b_{0} d b_{1}= & a_{0}\left(d a_{1} \cdot b_{0}\right) \cup_{1} d b_{1} \\
= & a_{0}\left(d\left(a_{1} b_{0}\right)-a_{1} d b_{0}\right) \cup_{1} d b_{1} \\
= & a_{0}\left[d\left(a_{1} b_{0}\right) \cup_{1} d b_{1}\right]-a_{0} a_{1}\left(d b_{0} \cup_{1} d b_{1}\right) \\
= & a_{0}\left[d\left(a_{1} b_{0} b_{1}\right)-b_{1} d\left(a_{1} b_{0}\right)-a_{1} b_{0} d b_{1}\right]-a_{0} a_{1}\left[d\left(b_{0} b_{1}\right)-b_{1} d b_{0}-b_{0} d b_{1}\right] \\
= & a_{0} d\left(a_{1} b_{0} b_{1}\right)-a_{0} b_{1} d\left(a_{1} b_{0}\right)-a_{0} a_{1} b_{0} d b_{1}-a_{0} a_{1} d\left(b_{0} b_{1}\right) \\
& +a_{0} a_{1} b_{1} d b_{0}+a_{0} a_{1} b_{0} d b_{1},
\end{aligned}
$$

and we are done.
5.4. Non-commutative differential forms as cup-one dgas. Let $A$ be a commutative $R$-algebra, and let $(\Omega(A), d)$ be the dga of non-commutative differential forms on $A$. Note that $d u \in D(\Omega(A))$ for all $u \in \Omega^{1}(A)$.

Theorem 5.3. The differential graded algebra $(\Omega(A), d)$ is a $\cup_{1}-d g a$.
Proof. The algebra $\Omega(A)$ inherits from $T(A)$ the structure of a graded algebra with cupone products. Furthermore, the Steenrod identity (3.9) holds in the dga $(\Omega(A), d)$.
We define an $R$-linear map $\circ: T^{2}(A) \otimes_{R} T^{2}(A) \rightarrow T^{2}(A)$ by

$$
\begin{equation*}
\left(a_{0} \otimes a_{1} \otimes a_{2}\right) \circ\left(b_{0} \otimes b_{1} \otimes b_{2}\right)=a_{0} b_{0} \otimes a_{1} b_{1} \otimes a_{2} b_{2} \tag{5.11}
\end{equation*}
$$

It is clear that this map induces a map $\circ: \Omega^{2}(A) \otimes_{R} \Omega^{2}(A) \rightarrow \Omega^{2}(A)$. A straightforward computation using equation (5.1) for the cup product and equation (5.7) for the cup-one product shows that $\left(u_{1} \cup u_{2}\right) \circ\left(v_{1} \cup v_{2}\right)=\left(u_{1} \cup_{1} v_{1}\right) \cup\left(u_{2} \cup v_{2}\right)$ for all $u_{1}, u_{2}, v_{1}, v_{2} \in \Omega^{1}(A)$
and thus the map $\circ$ restricts to a well-defined map $\circ: D^{2}(\Omega(A)) \otimes_{R} D^{2}(\Omega(A)) \rightarrow D^{2}(\Omega(A))$ which obeys formula (3.6).
In view of Lemma 3.8, we are left with showing that equation (3.8) holds for any elements $u, v, w \in \Omega^{1}(A)$ with $d u \in D^{2}(\Omega(A))$. As in the proof of Theorem 4.4, we will show that

$$
\begin{equation*}
u \cup_{1}(v \cup w)=-\left(u \cup_{1} v\right) \cup w+\sum_{i}\left(u_{1, i} \cup_{1} v\right) \cup\left(u_{2, i} \cup_{1} w\right)-v \cup\left(u \cup_{1} w\right) \tag{5.12}
\end{equation*}
$$

where $d u=\sum_{i} u_{1, i} \cup u_{2, i}$.
From the definition of $\Omega(A)$, it is sufficient to show that equation (5.12) holds in $T^{*}(A)$ via the canonical projection $J: T^{*}(A) \rightarrow \Omega^{*}(A)$. For this we can identify $u, v, w$ with the following preimages under this projection: $u=a_{0} d a_{1}=a_{0} \otimes a_{1}-a_{0} a_{1} \otimes 1, v=b_{0} \otimes b_{1}$, and $w=c_{0} \otimes c_{1}$, respectively.
Then using the formula $d\left(a_{0} d a_{1}\right)=d a_{0} \cup d a_{1}$ and the formulas for the cup product and cup-one product in $T^{*}(A)$, it follows that

$$
\begin{aligned}
u \cup_{1}(v \cup w)= & -\left(a_{0} b_{0} \otimes b_{1} c_{0} \otimes c_{1} a_{1}\right)+\left(a_{0} a_{1} b_{0} \otimes b_{1} c_{0} \otimes c_{1}\right) \\
-\left(u \cup_{1} v\right) \cup w= & -a_{0} b_{0} \otimes a_{1} b_{1} c_{0} \otimes c_{1}+a_{0} a_{1} b_{0} \otimes b_{1} c_{0} \otimes c_{1} \\
\left(d a_{0} \cup_{1} v\right) \cup\left(d a_{1} \cup_{1} w\right)= & b_{0} \otimes a_{0} b_{1} c_{0} \otimes a_{1} c_{1}-a_{0} b_{0} \otimes b_{1} c_{0} \otimes a_{1} c_{1} \\
& +a_{0} b_{0} \otimes a_{1} b_{1} c_{0} \otimes c_{1}-b_{0} \otimes a_{0} a_{1} b_{1} c_{0} \otimes c_{1} \\
-v \cup\left(u \cup_{1} w\right)= & -b_{0} \otimes a_{0} b_{1} c_{0} \otimes a_{1} c_{1}+b_{0} \otimes a_{0} a_{1} b_{1} c_{0} \otimes c_{1} .
\end{aligned}
$$

Equation (5.12) follows and the proof is complete.
The following example is analogous to an example in [1] to indicate that the naive righthanded analogue of the Hirsch identity (3.3) does not hold in this context.
Example 5.4. As in the proof of Theorem 5.3, let $u=a_{0} \otimes a_{1}-a_{0} a_{1} \otimes 1, v=b_{0} \otimes b_{1}$, and $w=c_{0} \otimes c_{1}$. Then

$$
\begin{aligned}
u \cup_{1}(v \cup w) & =-\left(a_{0} b_{0} \otimes b_{1} c_{0} \otimes c_{1} a_{1}\right)+\left(a_{0} a_{1} b_{0} \otimes b_{1} c_{0} \otimes c_{1}\right) \\
\left(u \cup_{1} v\right) \cup w & =\left(a_{0} b_{0} \otimes a_{1} b_{1}-a_{0} a_{1} b_{0} \otimes b_{1}\right) \cup\left(c_{0} \otimes c_{1}\right) \\
& =a_{0} b_{0} \otimes a_{1} b_{1} c_{0} \otimes c_{1}-a_{0} a_{1} b_{0} \otimes b_{1} c_{0} \otimes c_{1} \\
v \cup\left(u \cup_{1} w\right) & =\left(b_{0} \otimes b_{1}\right) \cup\left(a_{0} c_{0} \otimes a_{1} c_{1}-a_{0} a_{1} c_{0} \otimes c_{1}\right) \\
& =b_{0} \otimes b_{1} a_{0} c_{0} \otimes a_{1} c_{1}-b_{0} \otimes b_{1} a_{0} a_{1} c_{0} \otimes c_{1},
\end{aligned}
$$

and so

$$
u \cup_{1}(v \cup w) \neq\left(u \cup_{1} v\right) \cup w+v \cup\left(u \cup_{1} w\right) .
$$

Remark 5.5. The referee has suggested there may be a unified approach towards showing that our two main examples-cochain algebras and non-commutative differential
forms-are indeed $\cup_{1}$-dgas. This may rely on the notion of G -algebra in the sense of Kadeishvili [18] and use recent work of Battikh and Issaoui [4].

## 6. Binomial cup-one algebras

In this section we review the notions of binomial ring and ring of integrally-valued polynomials, and define binomial cup-one algebras.
6.1. Binomial rings. Following P. Hall [15], Wilkerson [36], Elliott [9], and D. Yau [38], we make the following definition.

Definition 6.1. A commutative, unital ring $A$ is a binomial ring if $A$ is torsion-free (as a $\mathbb{Z}$-module), and the elements

$$
\begin{equation*}
\binom{a}{n}:=a(a-1) \cdots(a-n+1) / n!\in A \otimes_{\mathbb{Z}} \mathbb{Q} \tag{6.1}
\end{equation*}
$$

lie in $A$ for every $a \in A$ and every $n>0$.
For instance, the ring of integers $\mathbb{Z}$ is a binomial ring.
As shown by Xantcha in [37], building on work of Ekedahl from [8], this condition is equivalent to the existence of maps $\zeta_{n}: A \rightarrow A, a \mapsto\binom{a}{n}$ for all integers $n \geq 0$ such that the following five axioms are satisfied:

$$
\begin{align*}
\zeta_{n}(a+b) & =\sum_{i+j=n} \zeta_{i}(a) \zeta_{j}(b),  \tag{6.2}\\
\zeta_{n}(a b) & =\sum_{m=0}^{n} \zeta_{m}(a) \cdot\left(\sum_{\substack{q_{1}+\cdots+q_{m}=n \\
q_{i}=1}} \zeta_{q_{1}}(b) \cdots \zeta_{q_{m}}(b)\right)  \tag{6.3}\\
\zeta_{m}(a) \zeta_{n}(a) & =\sum_{k=0}^{n}\binom{m+k}{n}\binom{n}{k} \zeta_{m+k}(a)  \tag{6.4}\\
\zeta_{n}(1) & =0 \quad \text { for } n \geq 2  \tag{6.5}\\
\zeta_{0}(a) & =1 \quad \text { and } \quad \zeta_{1}(a)=a \tag{6.6}
\end{align*}
$$

Remark 6.2. As shown in [9, 36], a torsion-free ring $A$ is a binomial ring if and only if $A$ is a $\lambda$-ring and all its Adams operations are the identity.

The following lemma shows that if $A$ is a commutative ring, possibly without unit, that satisfies the assumptions of Definition 6.1, then there is a natural way to extend $A$ to a binomial ring by adding a unit.

Lemma 6.3. Let A be a commutative ring, possibly without unit. Suppose $A$ is a module over a binomial ring $R$, and suppose $\binom{a}{n} \in A$ for every $a \in A$ and $n>0$. Then $R \oplus A$ has the natural structure of a binomial ring with unit $1 \in R$, where $R$ is viewed as a direct summand of $R \oplus A$.

Proof. For $a \in A$ and $n>0$, set $P_{n}(a)$ equal to $P_{n}(a)=a(a-1) \cdots(a-n+1)$. Due to our assumption, equation (6.2) is equivalent to the statement that for $n>0$, the equation

$$
\begin{equation*}
P_{n}(a+b)=P_{n}(a)+\sum_{i=1}^{n-1}\binom{n}{i} P_{i}(a) P_{n-i}(b)+P_{n}(b) \tag{6.7}
\end{equation*}
$$

holds for all $a, b \in A$.
Define the multiplication on $R \oplus A$ by setting $(r+a)(s+b)=r s+r b+a s+a b$ where $r s$ is the mutiplication in $R$, the products $r b$ and as come from the structure of $A$ as an $R$ module, and $a b$ denotes the product in $A$. Since $a s=s a$ and $r b=b r$ for all $r, s \in R$ and $a, b \in A$, this defines a commutative ring structure on the $R$-module $R \oplus A$, with unit as advertised.
Dividing now both sides in equation (6.7) by $n!$ and using the identity $\binom{n}{i}=\frac{n!}{i!(n-i)!}$, it follows that in $(R \oplus A) \otimes_{\mathbb{Z}} \mathbb{Q}$ we have

$$
\begin{aligned}
\zeta_{n}(r+a)=\frac{P_{n}(r+a)}{n!} & =\frac{P_{n}(r)}{n!}+\sum_{i+1}^{n-1}\binom{n}{i} \frac{P_{i}(r) P_{n-i}(a)}{n!}+\frac{P_{n}(a)}{n!} \\
& =\frac{P_{n}(r)}{n!}+\sum_{i+1}^{n-1} \frac{P_{i}(r) P_{n-i}(a)}{i!(n-i)!}+\frac{P_{n}(a)}{n!} \\
& =\zeta_{n}(r)+\sum_{i=1}^{n-1} \zeta_{i}(r) \zeta_{n-i}(a)+\zeta_{n}(a),
\end{aligned}
$$

and so $\zeta_{n}(r+a) \in A$, for every $r \in R, a \in A$, and $n>0$. Hence, $R \oplus A$ is a binomial ring.
6.2. $R$-valued polynomials. Suppose $R$ is an integral domain, and let $K=\operatorname{Frac}(R)$ be its field of fractions. Let $K[X]$ be the ring of polynomials in a set of formal variables $X$, with coefficients in $K$, and let $R^{X}$ be the free $R$-module on the set $X$. Following [5, 9, 10], we define the ring of $R$-valued polynomials (in the variables $X$ and with coefficients in $K)$ as the subring

$$
\begin{equation*}
\operatorname{Int}\left(R^{X}\right):=\left\{q \in K[X] \mid q\left(R^{X}\right) \subseteq R\right\} . \tag{6.8}
\end{equation*}
$$

Assume now that the domain $R$ has characteristic 0 , that is, $R$ is torsion-free as an abelian group. Then $\operatorname{Int}\left(R^{X}\right)$ is a binomial ring, which satisfies a type of universality property
that makes it into the free binomial ring on variables in $X$. As a consequence (at least when $R=\mathbb{Z}$ ), any binomial ring is a quotient of $\operatorname{Int}\left(R^{X}\right)$, for some set $X$.

As shown in [9, Theorem 7.1], every torsion-free ring $A$ is contained in a smallest binomial ring, $\operatorname{Bin}(A)$, which is defined as the intersection of all binomial subrings of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ containing $A$. Alternatively,

$$
\begin{equation*}
\operatorname{Bin}(A)=\operatorname{Int}\left(\mathbb{Z}^{X_{A}}\right) /\left(I_{A} \mathbb{Q}^{X_{A}} \cap \operatorname{Int}\left(\mathbb{Z}^{X_{A}}\right)\right), \tag{6.9}
\end{equation*}
$$

where $\mathbb{Z}\left[X_{A}\right]$ denotes the polynomial ring in variables indexed by the elements of $A$, and $I_{A}$ is the kernel of the canonical epimorphism $\mathbb{Z}\left[X_{A}\right] \rightarrow A$. Moreover, if $A$ is generated as a $\mathbb{Z}$-algebra by a collection of elements $\left\{a_{i}\right\}_{i \in J}$, then $\operatorname{Bin}(A)$ is the $\mathbb{Z}$-subalgebra of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by all elements of the form $\zeta_{n}\left(a_{i}\right)=\binom{a_{i}}{n}$ with $i \in J$ and $n \geq 0$. Finally, it is readily seen that $\operatorname{Bin}(\mathbb{Z}[X])=\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.
6.3. Products of binomial polynomials. Let $I: X \rightarrow \mathbb{Z}_{\geq 0}$ be a function which takes only finitely many non-zero values; in other words, the support of the function, $\operatorname{supp}(I):=$ $\{x \in X \mid I(x) \neq 0\}$, is a finite subset of $X$. Given a coefficient ring $R$, we associate to such a function an $R$-valued polynomial function, as follows.

First let $\mathbf{x}=\left\{x_{1}, \ldots, x_{n}\right\}$ be a finite subset of $X$ that contains $\operatorname{supp}(I)$. Then the indexing function $I: X \rightarrow \mathbb{Z}_{\geq 0}$ takes values $I\left(x_{k}\right)=i_{k}$ for $k=1, \ldots, n$, and 0 otherwise, and so it may be identified with the $n$-tuple $\left(i_{1}, \ldots, i_{n}\right) \in\left(\mathbb{Z}_{\geq 0}\right)^{n}$. We define a polynomial, $\zeta_{I}(\mathbf{x})$, in the variables $x_{1}, \ldots, x_{n}$, by

$$
\begin{equation*}
\zeta_{I}(\mathbf{x})=\prod_{k=1}^{n} \zeta_{i_{k}}\left(x_{k}\right) \tag{6.10}
\end{equation*}
$$

Clearly, $\zeta_{I}(\mathbf{x})$ is an $R$-valued polynomial in $\operatorname{Int}\left(R^{\mathbf{x}}\right) \subset \operatorname{Int}\left(R^{X}\right)$. That is to say, given an $n$-tuple $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right) \in R^{\mathbf{x}}$, the evaluation $\zeta_{I}(\mathbf{a}):=\zeta_{I}(\mathbf{x})(\mathbf{a})$ of the polynomial $\zeta_{I}(\mathbf{x})$ at $x_{k}=a_{k}$ is an element of $R$.
We now define an $R$-valued polynomial $\zeta_{I} \in \operatorname{Int}\left(R^{X}\right)$ by setting $\zeta_{I}=\zeta_{I}(\mathbf{x})$, for some finite set of variables $\mathbf{x}$ with $\mathbf{x} \supseteq \operatorname{supp}(I)$. Since $\zeta_{0}(a)=1$ for all $a \in R$, this definition is independent of the choice of $\mathbf{x}$. Given any $\mathbf{a} \in R^{X}$, we have a well-defined evaluation $\zeta_{I}(\mathbf{a}) \in R$; in fact, we do have such an evaluation for any function $\mathbf{a}: X \rightarrow R$, again since $I$ has finite support. For instance, if $I=\mathbf{0}$ is the function that takes only the value 0 , then $\zeta_{0}$ is the constant polynomial 1 in the variables $X$, and $\zeta_{0}(\mathbf{a})=1$, for any $\mathbf{a}: X \rightarrow R$.
The above notions extend to an arbitrary binomial ring $A$. For instance, if $I=\left(i_{1}, \ldots, i_{n}\right)$ is an $n$-tuple of non-negative integers, the evaluation of the polynomial function $\zeta_{I}(\mathbf{x})$ from (6.10) at an $n$-tuple $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ of elements in $A$ is equal to $\zeta_{I}(\mathbf{a})=\prod_{k=1}^{n} \zeta_{i_{k}}\left(a_{k}\right)$. More generally, the evaluation $\zeta_{I}(\mathbf{a}) \in A$ is defined for any function $\mathbf{a}: X \rightarrow A$.
6.4. A basis for integer-valued polynomials. We restrict now to the case when $R=\mathbb{Z}$. The next theorem provides a very useful $\mathbb{Z}$-basis for the $\operatorname{ring} \operatorname{Int}\left(\mathbb{Z}^{X}\right)$ of integer-valued polynomials; for a proof, we refer to [5, Proposition XI.I.12] and [9, Lemma 2.2].
Theorem 6.4. The $\mathbb{Z}$-module $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ is free, with basis consisting of all polynomials of the form $\zeta_{I}$ with $I: X \rightarrow \mathbb{Z}_{\geq 0}$ a function with finite support.

Alternatively, one may take as a basis for $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ all polynomials $\zeta_{I}(\mathbf{x})$ with $\operatorname{supp}(I)=$ $\mathbf{x}$, together with the constant polynomial $\zeta_{\mathbf{0}}$. We emphasize that in the products $\zeta_{I}(\mathbf{x})$, there is no repetition allowed among the variables comprising the set $\mathbf{x}$. For instance, the product $\zeta_{m}(x) \zeta_{n}(x)$ is not part of the aforementioned $\mathbb{Z}$-basis; rather, formula (6.4) expresses it as a linear combination of the binomials $\zeta_{m}(x), \ldots, \zeta_{m+n}(x)$. On the other hand, if $I$ and $J$ have disjoint supports, we have that $\zeta_{I} \cdot \zeta_{J}=\zeta_{I+J}$, and this polynomial is again part of the aforementioned basis for $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.
Example 6.5 (G. Pólya). Every degree $n$ integer-valued polynomial $f$ in a single variable $x$ can be written uniquely as a linear combination, $f(x)=\sum_{k=0}^{n} c_{k}\binom{x}{k}$, where the coefficients $c_{k} \in \mathbb{Z}$ are defined recursively by $c_{0}=f(0)$ and $c_{k}=f(k)-\sum_{i=0}^{k-1} c_{i}\binom{k}{i}$.

As an application of the above theorem, we obtain the following universality property for free binomial rings.

Corollary 6.6. Let $X$ be a set, let A be a binomial ring, and let $\phi: X \rightarrow A$ be a map of sets. There is then a unique extension of $\phi$ to a map $\tilde{\phi}: \operatorname{Int}\left(\mathbb{Z}^{X}\right) \rightarrow A$ of binomial rings.

Proof. For each finitely supported function $I: X \rightarrow \mathbb{Z}_{\geq 0}$ and each finite subset $\mathbf{x}=$ $\left\{x_{1}, \ldots, x_{n}\right\}$ of $X$, we put

$$
\begin{equation*}
\tilde{\phi}\left(\zeta_{I}(\mathbf{x})\right)=\zeta_{I}\left(\phi\left(x_{1}\right), \ldots, \phi\left(x_{n}\right)\right) \tag{6.11}
\end{equation*}
$$

where on the right side we are using the $\zeta$-maps of $A$. In view of Theorem 6.4, this formula defines a $\mathbb{Z}$-linear map, $\tilde{\phi}: \operatorname{Int}\left(\mathbb{Z}^{X}\right) \rightarrow A$. Since $\zeta_{1}(x)=x$ for all $x \in X$, the map $\tilde{\phi}$ agrees with $\phi$ on $X$. Using equation (6.4), it is readily verified that the map $\tilde{\phi}$ respects the multiplicative structures on source and target. Moreover, by construction, $\tilde{\phi}$ preserves the binomial structures. Therefore, $\tilde{\phi}$ is a morphism of binomial rings.

A characterization of binomial rings in terms of integer-valued polynomials is given in the following theorem (see [9, Theorem 4.1] and [38, Theorem 5.34]).

Theorem 6.7. A ring $R$ is a binomial ring if and only if the following two conditions hold:
(1) $R$ is $\mathbb{Z}$-torsion-free,
(2) $R$ is the homomorphic image of a ring $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ of integer-valued polynomials for some set $X$.

Corollary 6.8. Let $R_{1}$ and $R_{2}$ be binomial rings. Then the tensor product $R_{1} \otimes_{\mathbb{Z}} R_{2}$, with product $(a \otimes b) \cdot(c \otimes d)=a c \otimes b d$, is a binomial ring.

Proof. By Theorem 6.7, it suffices to show that $R_{1} \otimes_{\mathbb{Z}} R_{2}$ is $\mathbb{Z}$-torsion-free and the homomorphic image of $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ for some set $X$. The abelian group $R_{1} \otimes_{\mathbb{Z}} R_{2}$ is torsion-free, since both $R_{1}$ and $R_{2}$ are torsion-free as $\mathbb{Z}$-modules. If we have ring epimorphisms $\operatorname{Int}\left(X_{i}\right) \rightarrow R_{i}$ for $i=1,2$, then $R_{1} \otimes_{\mathbb{Z}} R_{2}$ is a homomorphic image of $\operatorname{Int}\left(X_{1}\right) \otimes_{\mathbb{Z}} \operatorname{Int}\left(X_{2}\right)$. The result now follows, since $\operatorname{Int}\left(X_{1}\right) \otimes_{\mathbb{Z}} \operatorname{Int}\left(X_{2}\right) \cong \operatorname{Int}\left(X_{1} \cup X_{2}\right)$ by Theorem 6.4.

One can also prove Corollay 6.8 by considering $R_{1} \otimes_{\mathbb{Z}} R_{2}$ as a subring of $R_{1} \otimes_{\mathbb{Z}} R_{2} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the $\zeta_{n}$ maps have well-defined restrictions to $R_{1} \otimes_{\mathbb{Z}} R_{2}$.
6.5. Binomial cup-one algebras. We now combine the notions of graded algebras with cup-one products (§3.2) and cup-one algebras (§3.3) on one hand, with the notion of binomial rings (§6.1) on the other hand.
Definition 6.9. Let $R$ be a binomial ring. A graded $R$-algebra $A$ is called a binomial graded algebra with cup-one products if the following conditions are satisfied.
(i) $A$ is a graded algebra with cup-one products.
(ii) The $R$-submodule $R \oplus A^{1} \subset A^{\leq 1}$, with multiplication $A^{1} \otimes_{R} A^{1} \rightarrow A^{1}$ given by the cup-one product, is a binomial ring.

The binomial ring structure from part (ii) is the one described in Lemma 6.3. In concrete terms, the ring structure on $R \oplus A^{1}$ is given by $(r+a)(s+b)=r s+\left(r b+s a+a \cup_{1} b\right)$, for $r, s \in R$ and $a, b \in A^{1}$. Denoting the binomial operations in $R$ by $\zeta_{n}^{R}$ and setting

$$
\begin{equation*}
\zeta_{n}^{A}(a):=a \cup_{1}(a-1) \cup_{1} \cdots \cup_{1}(a-n+1) / n! \tag{6.12}
\end{equation*}
$$

for $a \in A^{1}$ and $n \in \mathbb{N}$, with the convention that $\zeta_{0}^{A}(a)=1 \in R$, the binomial operations in $R \oplus A^{1}$ are given by $\zeta_{n}(r+a)=\sum_{i+j=n} \zeta_{i}^{R}(r) \zeta_{j}^{A}(a)$.
Definition 6.10. A differential graded $R$-algebra $(A, d)$ is called a binomial $\cup_{1}-d g a$ if it is both a $\cup_{1}$-dga and a binomial graded algebra with cup-one products.

If $\varphi: A \rightarrow B$ is a morphism of binomial $\cup_{1}$-dgas, it follows from the definitions that $\varphi\left(\zeta_{n}(a)\right)=\zeta_{n}(\varphi(a))$, for all $n \geq 1$ and all $a \in A^{1}$.
Lemma 6.11. In a binomial $\cup_{1}-d g a(A, d)$, the following identities hold for $a, b, c \in A^{1}$,

$$
\begin{align*}
(a \cup b) \cup_{1} c & =a \cup\left(b \cup_{1} c\right)+\left(a \cup_{1} c\right) \cup b,  \tag{6.13}\\
d\left(a \cup_{1} b\right) & =-a \cup b-b \cup a+(d a) \cup_{1} b,  \tag{6.14}\\
\zeta_{n+1}(a) & =\frac{\zeta_{n}(a) \cup_{1} a-n \zeta_{n}(a)}{n+1} \text { for } n \geq 1, \tag{6.15}
\end{align*}
$$

where in (6.14) we are assuming $d a \in D^{2}(A)$ and $d b=0$.

Proof. Equation (6.13) is the left-side Hirsch identity from (3.4); equation (6.14) is the $\cup_{1}-d$ formula (3.7) in the case $d b=0$; finally, equation (6.15) follows straight from equation (6.1), or from equation (6.4) in the case $m=k+1$ and $n=1$.

For simplicity, we shall at times abuse notation and write formula (6.15) as $\zeta_{n+1}(a)=$ $\frac{1}{n+1}\left(\zeta_{n}(a) \cup_{1}(a-n)\right)$.
6.6. Differentiating the zeta maps. The goal for the rest of this section is to derive a formula for the differential of $\zeta_{n}(a)$ for $a$ a cocycle in $A^{1}$ and $n \geq 2$, in the setting where $A$ is a binomial $\cup_{1}$-dga. We start with the case $n=2$.

Lemma 6.12. Let $A$ be a binomial $\cup_{1}-d g a$. If $a \in Z^{1}(A)$, then $d \zeta_{2}(a)=-a \cup a$.

Proof. By assumption, we have a map $\zeta_{2}: A^{1} \rightarrow A^{1}$ given by $\zeta_{2}(a)=\frac{1}{2}\left(a \cup_{1} a-a\right)$ for all $a \in A^{1}$. If $d a=0$, then $d\left(a \cup_{1} a\right)=-2 a \cup a$, by (6.14). Hence,

$$
d \zeta_{2}(a)=\frac{1}{2} d\left(a \cup_{1} a-a\right)=-\frac{1}{2}(2 a \cup a)=-a \cup a,
$$

and we are done.

In greater generality, we have the following result relating the cup product, the differential, and the $\zeta$-maps in a binomial $\cup_{1}$-dga.

Theorem 6.13. Let $A$ be a binomial $\cup_{1}-d g a$. Then, for each $a \in Z^{1}(A)$ and each $k \geq 0$, we have

$$
\begin{equation*}
d \zeta_{k}(a)=-\sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \cup \zeta_{k-\ell}(a) \tag{6.16}
\end{equation*}
$$

More generally, if $I=\left(k_{1}, \ldots, k_{n}\right)$ with $k_{i} \geq 0$ and $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with $a_{i} \in Z^{1}(A)$, then

$$
\begin{equation*}
d\left(\zeta_{I}(\mathbf{a})\right)=-\sum_{\substack{I_{1}+I_{2}=I \\ I_{j} \neq 0}} \zeta_{I_{1}}(\mathbf{a}) \cup \zeta_{I_{2}}(\mathbf{a}) \tag{6.17}
\end{equation*}
$$

Proof. The first claim is proved by induction on $k$. Since $d a=0$, the case $k=1$ is vacuous, and the case $k=2$ is Lemma 6.12. Assume now that (6.16) holds for all $a$ in $Z^{1}(A)$. Working in $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$ and using the formulas listed in Lemma 6.11,
we get

$$
\begin{aligned}
& d \zeta_{k+1}(a) \underset{(6.15)}{=} \frac{1}{k+1} d\left(\zeta_{k}(a) \cup_{1} a\right)-\frac{k}{k+1} d \zeta_{k}(a) \\
& \underset{\substack{(6.14) \\
(6.16)}}{=}-\frac{1}{k+1}\left(\zeta_{k}(a) \cup a+a \cup \zeta_{k}(a)-d \zeta_{k}(a) \cup_{1} a\right)+\frac{k}{k+1} \sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \cup \zeta_{k-\ell}(a) \\
& \underset{(6.16)}{=}-\frac{1}{k+1}\left(\zeta_{k}(a) \cup a+a \cup \zeta_{k}(a)+\sum_{\ell=1}^{k-1}\left(\zeta_{\ell}(a) \cup \zeta_{k-\ell}(a)\right) \cup_{1} a\right)+\frac{k}{k+1} \sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \cup \zeta_{k-\ell}(a) \\
& \underset{(6.13)}{=}-\frac{1}{k+1}\left(\zeta_{k}(a) \cup a+a \cup \zeta_{k}(a)+\sum_{\ell=1}^{k-1}\left(\zeta_{\ell}(a) \cup\left(\zeta_{k-\ell}(a) \cup_{1} a\right)+\left(\zeta_{\ell}(a) \cup_{1} a\right) \cup \zeta_{k-\ell}(a)\right)\right) \\
& +\frac{k}{k+1} \sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \cup \zeta_{k-\ell}(a) \\
& =-\frac{1}{k+1}\left(\zeta_{k}(a) \cup a+a \cup \zeta_{k}(a)+\right. \\
& \left.\sum_{\ell=1}^{k-1}\left(\zeta_{\ell}(a) \cup\left(\zeta_{k-\ell}(a) \cup_{1}(a-k+\ell)\right)+\left(\zeta_{\ell}(a) \cup_{1}(a-\ell)\right) \cup \zeta_{k-\ell}(a)\right)\right) \\
& \underset{(6.15)}{=}-\frac{1}{k+1}\left(\zeta_{k}(a) \cup a+a \cup \zeta_{k}(a)+\right. \\
& \left.\sum_{\ell=1}^{k-1}\left((k-\ell+1) \cdot \zeta_{\ell}(a) \cup \zeta_{k-\ell+1}(a)+(\ell+1) \zeta_{\ell+1}(a) \cup \zeta_{k-\ell}(a)\right)\right) \\
& =-\frac{1}{k+1} \sum_{\ell=1}^{k}\left((k-\ell+1) \cdot \zeta_{\ell}(a) \cup \zeta_{k-\ell+1}(a)+\ell \zeta_{\ell}(a) \cup \zeta_{k-\ell+1}(a)\right) \\
& =-\sum_{\ell=1}^{k} \zeta_{\ell}(a) \cup \zeta_{k-\ell+1}(a) \text {. }
\end{aligned}
$$

To prove the second claim, we need to show the following: Given $a_{1}, \ldots, a_{n} \in Z^{1}(A)$ and $k_{1}, \ldots, k_{n} \in \mathbb{Z}_{\geq 0}$, we have

$$
\begin{align*}
d\left(\zeta_{k_{1}}\left(a_{1}\right) \cup_{1}\right. & \left.\cdots \cup_{1} \zeta_{k_{n}}\left(a_{n}\right)\right)= \\
& -\sum_{\ell_{1}, \ldots, \ell_{n}}\left(\zeta_{\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{\ell_{n}}\left(a_{n}\right)\right) \cup\left(\zeta_{k_{1}-\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n}-\ell_{n}}\left(a_{n}\right)\right), \tag{6.18}
\end{align*}
$$

where the sum is over all $n$-tuples, $\left(\ell_{1}, \ldots, \ell_{n}\right)$, of non-negative integers such that $\ell_{i} \leq k_{i}$, with the tuples $(0, \ldots, 0)$ and $\left(k_{1}, \ldots, k_{n}\right)$ excluded. In particular, if we set $a=\zeta_{k_{1}}\left(a_{1}\right) \cup_{1}$ $\cdots \cup_{1} \zeta_{k_{n}}\left(a_{n}\right)$ we have that $d a \in D^{2}(A)$, as needed in order to apply formula (6.14).

The above claim is established by induction on $n$, the base case $n=1$ being (6.16), which has just been verified. Assume (6.18) holds for $b:=\zeta_{k_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}}\left(a_{n-1}\right)$, where not all $k_{i}$ 's are equal to 0 . Then, for all $a_{n} \in Z^{1}(A)$,

$$
\begin{aligned}
& d\left(b \cup_{1} a_{n}\right) \underset{(6.14)}{=}-b \cup a_{n}-a_{n} \cup b+(d a) \cup_{1} a_{n} \\
& \underset{(6.18)_{n-1}}{=}-b \cup a_{n}-a_{n} \cup b-\left(\sum_{\ell_{1}, \ldots, \ell_{n-1}}\left(\zeta_{\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{\ell_{n-1}}\left(a_{n-1}\right)\right) \cup\right. \\
& \left.\quad\left(\zeta_{k_{1}-\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}-\ell_{n-1}}\left(a_{n-1}\right)\right)\right) \cup_{1} a_{n} \\
& \underset{(6.13)}{=}-\zeta_{k_{1}}\left(a_{1}\right) \cup \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}}\left(a_{n-1}\right) \cup \zeta_{1}\left(a_{n}\right)-\zeta_{1}\left(a_{n}\right) \cup \zeta_{k_{1}}\left(a_{1}\right) \cup \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}}\left(a_{n-1}\right)- \\
& \quad \sum_{\ell_{1}, \ldots, \ell_{n-1}}\left(\left(\zeta_{\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{\ell_{n-1}}\left(a_{n-1}\right)\right) \cup\left(\zeta_{k_{1}-\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}-\ell_{n-1}}\left(a_{n-1}\right) \cup_{1} \zeta_{1}\left(a_{n}\right)\right)+\right. \\
& \left.\quad\left(\zeta_{\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{\ell_{n-1}}\left(a_{n-1}\right) \cup \cup_{1} \zeta_{1}\left(a_{n}\right)\right) \cup\left(\zeta_{k_{1}-\ell_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n-1}-\ell_{n-1}}\left(a_{n-1}\right)\right)\right),
\end{aligned}
$$

where not all $\ell_{i}$ 's are equal to 0 . Now set $\ell=k_{n}$. Rewriting this expression as a single sum yields formula (6.18) in the case when $\ell=1$. The general case follows by induction on $\ell$. Assuming equation (6.18) holds for $\ell=k_{n}$, we get

$$
\begin{aligned}
d\left(\zeta_{k_{1}}\left(a_{1}\right) \cup_{1} \cdots \cup_{1} \zeta_{k_{n}+1}\left(a_{n}\right)\right)= & d\left(b \cup_{1} \zeta_{\ell+1}\left(a_{n}\right)\right) \\
= & d\left(b \cup_{1}\left(\frac{1}{\ell+1} \zeta_{\ell}\left(a_{n}\right) \cup_{1} a_{n}-\frac{\ell}{\ell+1} \zeta_{\ell}\left(a_{n}\right)\right)\right) \\
= & \frac{1}{\ell+1} d\left(b \cup_{1} \zeta_{\ell}\left(a_{n}\right) \cup_{1} a_{n}\right)-\frac{\ell}{\ell+1} d\left(b \cup_{1} \zeta_{\ell}\left(a_{n}\right)\right) \\
= & \frac{1}{\ell+1}\left(-b \cup_{1} \zeta_{\ell}\left(a_{n}\right) \cup a_{n}-a_{n} \cup b \cup_{1} \zeta_{\ell}\left(a_{n}\right)+\right. \\
& \left.d\left(b \cup_{1} \zeta_{\ell}\left(a_{n}\right)\right) \cup_{1} a_{n}\right)-\frac{\ell}{\ell+1} d\left(b \cup_{1} \zeta_{\ell}\left(a_{n}\right)\right) .
\end{aligned}
$$

Using the induction hypothesis to express $d\left(b \cup_{1} \zeta_{\ell}\left(a_{n}\right)\right)$ as a sum according to (6.18), we obtain (after simplifying as before) equation (6.18), with $k_{n}$ replaced by $k_{n}+1$. This completes the proof.

In [27], we will provide a different proof of this theorem, in a more general setting, by dropping the assumption that $d a_{i}=0$ in formula (6.17).
6.7. Binomial operations on cochains and forms. We now show that cochain complexes with coefficients in a binomial ring are binomial $\cup_{1}$-dgas in a natural way.

Theorem 6.14. For any non-empty $\Delta$-complex $X$ and binomial ring $R$, the cochain algebra $R \oplus C^{\geq 1}(X ; R)$ is a binomial $\cup_{1}-d g a$.

Proof. By Theorem 4.4, the cochain algebra $C^{\geq 1}(X ; R)$ is a $\cup_{1}$-dga, with cup-one product on $A^{1}$ given by formula (4.7). We define maps $\zeta_{n}^{X}: C^{1}(X ; R) \rightarrow C^{1}(X ; R)$ by setting

$$
\begin{equation*}
\zeta_{n}^{X}(f)(e):=\zeta_{n}^{R}(f(e)) \quad \text { for } n \geq 1 \tag{6.19}
\end{equation*}
$$

for each 1-cochain $f \in C^{1}(X ; R)=\operatorname{Hom}\left(C_{1}(X ; R), R\right)$ and 1-simplex $e$ in $X$. Define $\zeta_{0}$ on $R \oplus C^{1}(X ; R)$ by setting $\zeta_{0}(f)=1 \in R$ for all $f \in C^{1}(X ; R)$ and $\zeta_{0}(r)=1$ for all $r \in R$. Then by Lemma 6.3 it follows that $R \oplus C^{1}(X ; R)$ is a binomial ring. This completes the proof.

Note that the evaluation $\zeta_{n}(f)(e)$ is equal to $\binom{f(e)}{n}$; in the case $R=\mathbb{Z}$, this is simply a binomial coefficient.

Remark 6.15. As noted in Example 4.2, the cup-product between 0-cocycles and 1cocycles need not be commutative, even when $X$ is a 1 -simplex. This explains why in Theorem 6.14 we had to replace $C^{0}(X ; R)$ by the ring $R$, so that the multiplication in $R \oplus C^{1}(X ; R)$ is commutative.

Remark 6.16. If $A=C^{*}(X ; \mathbb{Z})$ is the cochain algebra on a $\Delta$-complex $X$, then the $\zeta_{2}$ map defined in (6.19) equals minus the $\zeta$ map defined by Rybnikov in [31].

Example 6.17. For the cochain algebra $C=C^{*}(I ; \mathbb{Z})$ from Example 4.2, the $\zeta_{n}$-maps are given by $\zeta_{n}(m u)=\binom{m}{n} u$. In particular, $\zeta_{n}(u)=0$ for $n \geq 2$.

It is readily seen that the $\zeta$ maps of cochain algebras enjoy the following naturality property: If $h: X \rightarrow Y$ is a map of $\Delta$-complexes, then $\zeta_{n}^{X}\left(h^{\sharp}(f)\right)=\zeta_{n}^{Y}(f)$, for all $f \in C^{1}(Y ; R)$ and all $n \geq 0$.
6.8. Binomial operations on non-commutative forms. We conclude this section by showing that non-commutative forms on a binomial ring are also binomial $\cup_{1}$-dgas.

Theorem 6.18. If $A$ is a binomial ring over $R$, then $R \oplus \Omega^{\geq 1}(A)$ is a binomial $\cup_{1}-d g a$.
Proof. Let $T(A)$ be the tensor algebra from Proposition 5.1 and let $\Omega(A) \subset T(A)$ be the subalgebra with graded pieces defined in (5.3). Recall that $\Omega(A)$ inherits from $T(A)$ the structure of a graded algebra with cup-one products. Moreover, by Theorem 5.3, the algebra $\Omega^{*}(A)$ is a cup-one dga (over $\mathbb{Z}$ ).
Applying now Corollary 6.8 with $R_{1}=R_{2}=A$, we obtain in a natural fashion a binomial ring structure on $A \otimes_{\mathbb{Z}} A$. Observe now that the multiplication in $A \otimes_{\mathbb{Z}} A$ given in the corollary coincides with the cup-one product defined on $T^{1}(A)$. Hence, the $\mathbb{Z}$-submodule $\mathbb{Z} \oplus T^{1}(A) \subset T^{\leq 1}(A)$ acquires the structure of a commutative ring. Therefore, by Lemma $6.3, \mathbb{Z} \oplus \Omega^{1}(A)$ is also a binomial ring, and we are done.

## 7. Free cup-one algebras

In this section we define the notion of a free binomial cup-one dga, $\mathrm{T}(X)$, generated by a set, $X$. We show that maps of binomial graded algebras from $\mathrm{T}(X)$ to a binomial graded algebra $A$ are determined by the map restricted to the elements in $X$ and if $\mathrm{T}(X)$ and $A$ are binomial cup-one dgas then a map from $\mathrm{T}(X)$ to $A$ commutes with the differential if and only if it commutes with the differential on elements in $X$. We will work here over the ring $R=\mathbb{Z}$ and abbreviate $\otimes=\otimes_{\mathbb{Z}}$.
7.1. The free binomial graded algebra. Given a set $X$, we let $\mathrm{T}=\mathrm{T}(X)$ denote the free graded algebra with $\mathrm{T}^{1}$ equal to the ideal of all polynomials without constant term in the free binomial ring $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$. We define a cup-one map, $\cup_{1}:\left(T^{1} \otimes T^{1}\right) \otimes T^{1} \rightarrow T^{2}$, by means of the left Hirsch identity (3.4) and a map $\circ: T^{2} \otimes T^{2} \rightarrow T^{2}$ by means of equation (3.6). Then $\mathrm{T}(X)$ is a graded algebra with cup-one products called the free binomial graded algebra with cup-one products generated by $X$.
We now make this all more precise; we start with a definition. Recall from section 6.2 that $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ denotes the free binomial ring of integer-valued polynomials in variables from $X$.

Definition 7.1. The free binomial graded algebra on a set $X$, denoted $\mathrm{T}=\mathrm{T}(X)$, is the tensor algebra on $\mathfrak{m}_{X}$, the ideal of all integer-valued polynomials in variables $X$ without constant term,

$$
\begin{equation*}
\mathrm{T}^{*}(X)=T^{*}\left(\mathfrak{m}_{X}\right) . \tag{7.1}
\end{equation*}
$$

By construction, $\mathrm{T}^{0}(X)=\mathbb{Z}$ and $\mathrm{T}^{1}(X)=\mathfrak{m}_{X}$, and so $\mathrm{T}^{\leq 1}(X)=\mathrm{T}^{0} \oplus \mathrm{~T}^{1}$ is isomorphic to the free binomial ring $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$. By Theorem $6.4, T^{1}(X)$ is a free $\mathbb{Z}$-module, with basis consisting of all integer-valued polynomials of the form $\zeta_{I}=\zeta_{I}(\mathbf{x})$, where $I: X \rightarrow \mathbb{Z}_{\geq 0}$ has finite, non-empty support and $\mathbf{x}=\operatorname{supp}(I)$.
The $\mathbb{Z}$-module $\mathrm{T}^{1}$ comes endowed with a cup-one product map, $\mathrm{T}^{1} \otimes \mathrm{~T}^{1} \rightarrow \mathrm{~T}^{1}, a \otimes b \mapsto a b$. By analogy with the classical Hirsch formula for cochain algebras, we use this cup-one product to define a map $\mathrm{T}^{2} \otimes \mathrm{~T}^{1} \rightarrow \mathrm{~T}^{2}$ by

$$
\begin{equation*}
(a \otimes b) \otimes c \mapsto a c \otimes b+a \otimes b c . \tag{7.2}
\end{equation*}
$$

For the terms in the $U_{1}-d$ formula to be defined we include the map $\circ: T^{2} \otimes T^{2} \rightarrow T^{2}$ defined on basis elements by

$$
\begin{equation*}
\left(a_{1} \otimes a_{2}\right) \circ\left(b_{1} \otimes b_{2}\right)=\left(a_{1} b_{1}\right) \otimes\left(a_{2} b_{2}\right) . \tag{7.3}
\end{equation*}
$$

With this structure, $\mathrm{T}(X)$ is a graded algebra with cup-one products.
7.2. A binomial $\cup_{1}$-dga structure on $T(X)$. The next step is to show that the free binomial graded algebra $\mathrm{T}(X)$ admits a natural structure of a binomial $\cup_{1}$-dga.

Theorem 7.2. For any set $X$, the algebra $T=T(X)$ is a binomial $\cup_{1}$-dga, with differential $d_{\mathrm{\top}}$ satisfying $d_{\mathrm{\top}}(x)=0$ for all $x \in X$.

Proof. Our goal is to define a differential $d_{\mathrm{T}}: \mathrm{T} \rightarrow \mathrm{T}$ that satisfies the $\cup_{1}-d$ formula (3.7). We start by setting $d_{\mathrm{\top}}(x)=0$ for all $x \in X$. Next, for each basis element $\zeta_{I}=\zeta_{I}(\mathbf{x})$ of $\mathrm{T}^{1}(X)$ with $\operatorname{supp}(I)=\mathbf{x}$, we set

$$
\begin{equation*}
d_{\mathrm{T}}\left(\zeta_{I}(\mathbf{x})\right)=-\sum_{\substack{I_{1}+I_{2}=I \\ I_{j} \neq \boldsymbol{0}}} \zeta_{I_{1}}(\mathbf{x}) \otimes \zeta_{I_{2}}(\mathbf{x}) . \tag{7.4}
\end{equation*}
$$

This formula defines a $\mathbb{Z}$-linear map, $d_{\top}: \mathrm{T}^{1} \rightarrow \mathrm{~T}^{2}$. It remains to show that $d_{\mathrm{\top}}$ can be extended to a differential on the whole of T that satisfies the $\mathrm{U}_{1}-d$ formula.

To achieve this goal, we first define a 2-dimensional $\Delta$-complex, $\Delta(X)$, as follows. There is a single vertex; a 1 -simplex is a function $\mathbf{a}: X \rightarrow \mathbb{Z}$; finally, to each pair of 1 -simplices, $\mathbf{a}$ and $\mathbf{a}^{\prime}$, we assign a 2-simplex, (a, $\left.\mathbf{a}^{\prime}\right)$, with faces $\partial_{0}\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=\mathbf{a}^{\prime}, \partial_{1}\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=\mathbf{a}+\mathbf{a}^{\prime}$, and $\partial_{2}\left(\mathbf{a}, \mathbf{a}^{\prime}\right)=\mathbf{a}$. We let $C(X):=\left(C^{\bullet}(\Delta(X)), d_{\Delta}\right)$ denote the simplicial cochain algebra of $\Delta(X)$.
Next, we define a degree-preserving, $\mathbb{Z}$-linear $\operatorname{map} \varphi: \mathrm{T}^{\leq 2}(X) \rightarrow C(X)$, as follows. First define $\varphi: \mathrm{T}^{0}(X) \rightarrow C^{0}(X)$ by sending $1 \in \mathrm{~T}^{0}(X)=\mathbb{Z}$ to the the unit cochain $1 \in C^{0}(X)$. For each basis element $\zeta_{I}$ of $\mathrm{T}^{1}(X)$, we set $\varphi\left(\zeta_{I}\right) \in C^{1}(X)$ equal to the 1-cochain whose value on a 1-simplex $\mathbf{a}$ is $\zeta_{I}(\mathbf{a})$. Finally, we set $\varphi\left(\zeta_{I} \otimes \zeta_{J}\right) \in C^{2}(X)$ equal to the 2-cochain whose value on a 2 -simplex $\left(\mathbf{a}, \mathbf{a}^{\prime}\right)$ is $\zeta_{I}(\mathbf{a}) \cdot \zeta_{J}\left(\mathbf{a}^{\prime}\right)$. It follows directly from the definitions that $\varphi$ commutes with the cup and cup-one products. Comparing the expressions for the $\circ$ maps given in (4.11) and (7.3), we infer that $\varphi \circ\left(\circ_{T}\right)=\left(\circ_{\Delta}\right) \circ(\varphi \otimes \varphi)$. Moreover, since $d_{\Delta} \varphi(x)=0$ for all $x \in X$, Theorem 6.13 applies; comparing the expressions given by equations (6.18) and (6.17), we deduce that $\varphi \circ d_{\top}=d_{\Delta} \circ \varphi$.
We claim that $\varphi: \mathrm{T}^{\leq 2}(X) \rightarrow C(X)$ is a monomorphism. To prove the claim, first suppose that $\varphi\left(\sum \alpha_{I} \zeta_{I}\right)=0$, for some constants $\alpha_{I}$. Then for all 1-simplices, a, it follows that $\sum \alpha_{I} \zeta_{I}(\mathbf{a})=0$, and so $\sum \alpha_{I} \zeta_{I}$ is the zero polynomial. Since the polynomials $\zeta_{I}$ are elements in a basis for $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$, it follows that each $\alpha_{I}$ is equal to 0 . Therefore, $\varphi: \mathrm{T}^{1}(X) \rightarrow C^{1}(X)$ is a monomorphism.

Now suppose $\varphi\left(\sum \alpha_{I, J} \zeta_{I} \otimes \zeta_{J}\right)=0$, for some $\alpha_{I, J} \in \mathbb{Z}$. Then $\sum \alpha_{I, J} \zeta_{I}(\mathbf{a}) \cdot \zeta_{J}\left(\mathbf{a}^{\prime}\right)=0$, for all functions $\mathbf{a}, \mathbf{a}^{\prime}: X \rightarrow \mathbb{Z}$. Let $X^{\prime}$ be another (disjoint) copy of $X$, and for each $J$, let $J^{\prime}: X^{\prime} \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding indexing function. Viewing each $\zeta_{J^{\prime}}$ as a polynomial in $\operatorname{Int}\left(\mathbb{Z}^{X^{\prime}}\right)$, it follows that the polynomial $\sum \alpha_{I, J} \zeta_{I} \cdot \zeta_{J^{\prime}} \in \operatorname{Int}\left(\mathbb{Z}^{X \sqcup X^{\prime}}\right)$ is the zero polynomial. Note that, for each pair $I$ and $J$ of indexing functions, the functions $I$ and $J^{\prime}$ have disjoint supports; hence, $\zeta_{I} \cdot \zeta_{J^{\prime}}=\zeta_{K}$, where $\left.K\right|_{X}=I$ and $\left.K\right|_{X^{\prime}}=J^{\prime}$. Since these polynomials are
elements in a basis for $\operatorname{Int}\left(\mathbb{Z}^{X \sqcup X^{\prime}}\right)$, it follows that each $\alpha_{I, J}$ is equal to 0 , thus showing that $\varphi: \mathrm{T}^{2}(X) \rightarrow C^{2}(X)$ is a monomorphism.
This completes the argument that the map $\varphi: \mathrm{T}^{\leq 2}(X) \rightarrow C(X)$ is a monomorphism that commutes with the cup products, the cup-one products, the o maps, and the respective $d$ maps. By Theorem 6.14, the differential $d_{\Delta}$ satisfies the $\cup_{1}-d$ formula, and so it follows that $d_{\mathrm{T}}$ also satisfies the $\cup_{1}-d$ formula (3.7).
Next, we extend $d_{\mathrm{T}}$ to $\mathrm{T}=\mathrm{T}(X)$ using the graded Leibniz rule. To complete the proof, we need to show that $d_{\top} \circ d_{\top}=0$. It suffices to show this for the map $d_{\top} \circ d_{\top}: \mathrm{T}^{1} \rightarrow \mathrm{~T}^{3}$. By applying $d_{\top}$ to the first factors in the sum of tensor products from the right-hand side of (7.4), we have that

$$
\sum d_{\mathrm{T}}\left(\zeta_{I_{1}}(\mathbf{x})\right) \otimes \zeta_{I_{2}}(\mathbf{x})=\sum \zeta_{J_{1}}(\mathbf{x}) \otimes \zeta_{J_{2}}(\mathbf{x}) \otimes \zeta_{J_{3}}(\mathbf{x})
$$

where the sum is over all finitely-supported functions $J_{1}, J_{2}, J_{3}: X \rightarrow \mathbb{Z}_{\geq 0}$ with $J_{i} \neq \mathbf{0}$ and $J_{1}+J_{2}+J_{3}=I$. Similarly,

$$
\sum \zeta_{I_{1}}(\mathbf{x}) \otimes d_{\top}\left(\zeta_{I_{2}}(\mathbf{x})\right)=\sum \zeta_{J_{1}}(\mathbf{x}) \otimes \zeta_{J_{2}}(\mathbf{x}) \otimes \zeta_{J_{3}}(\mathbf{x})
$$

and using the graded Leibniz rule we have

$$
\begin{aligned}
d_{\mathrm{\top}} \circ d_{\mathrm{\top}}\left(\zeta_{I}(\mathbf{x})\right) & =-\sum d_{\mathrm{\top}}\left(\zeta_{I_{1}}(\mathbf{x})\right) \otimes \zeta_{I_{2}}(\mathbf{x})+\sum \zeta_{I_{1}}(\mathbf{x}) \otimes d_{\mathrm{\top}}\left(\zeta_{I_{2}}(\mathbf{x})\right) \\
& =\sum-\zeta_{J_{1}}(\mathbf{x}) \otimes \zeta_{J_{2}}(\mathbf{x}) \otimes \zeta_{J_{3}}(\mathbf{x})+\zeta_{J_{1}}(\mathbf{x}) \otimes \zeta_{J_{2}}(\mathbf{x}) \otimes \zeta_{J_{3}}(\mathbf{x}) \\
& =0 .
\end{aligned}
$$

This completes the proof.
Remark 7.3. Examples of $\cup_{1}-d$ dgas, $(\mathrm{T}(X), d)$ for which $d(x)$ is not zero for all $x \in X$ along with applications will be given in [27].
7.3. Maps with domain $T(X)$. We will show in [27] how to enhance the structure of the free binomial graded algebra on a set $X$ so as to construct a 1-minimal model over the integers for an arbitrary binomial $\cup_{1}$-dga.
As a stepping stone towards that goal, we show in the remainder of this section that maps from $\mathrm{T}(X)$ to a binomial graded algebra with cup-one products, $A$, are determined by maps of sets from $X$ to $A^{1}$ and in the case that $\mathrm{T}(X)$ and $A$ are binomial $\cup_{1}$-dgas, then a map of binomial $\cup_{1}$-dgas commutes with the respective differentials if and only if the map commutes with the differentials of the generators $x \in X$. We start with an extension lemma.
Lemma 7.4. Let $X$ be a set, let $A$ be a binomial graded $\mathbb{Z}$-algebra with cup-one products, and let $\phi: X \rightarrow A^{1}$ be a map of sets. There is then a unique extension of $\phi$ to a map $f: \mathrm{T}(X) \rightarrow A$ of binomial graded algebras.

Proof. Recall that $\mathrm{T}^{\leq 1}(X)$ is a graded ring, with $\mathrm{T}^{0}(X)=\mathbb{Z}$ the constant polynomials, and $\mathrm{T}^{1}(X)$ all integer-valued polynomials in variables $X$ with zero constant term. In view of Definition 6.9, the $\mathbb{Z}$-submodule $\mathbb{Z} \oplus A^{1} \subset A^{\leq 1}$, with multiplication $A^{1} \otimes A^{1} \rightarrow A^{1}$ given by the cup-one product, is a binomial ring.
The proof of Corollary 6.6 shows that the set map $\phi: X \rightarrow A^{1}$ extends to a map of binomial rings, $\tilde{\phi}: \mathbb{Z} \oplus \mathrm{T}^{1}(X) \rightarrow \mathbb{Z} \oplus A^{1}$ which is the identity in degree 0 . Finally, since $\mathrm{T}(X)$ is the free graded algebra generated by $\mathrm{T}^{1}(X)$, the restriction of $\tilde{\phi}$ to degree 1 pieces, $f: \mathrm{T}^{1}(X) \rightarrow A^{1}$, extends uniquely to a map $f: \mathrm{T}(X) \rightarrow A$ of binomial graded algebras with cup-one products.

Theorem 7.5. Let $X$ be a set, let $\left(\mathrm{T}(X), d_{\mathrm{T}}\right)$ and $\left(A, d_{A}\right)$ be binomial $\cup_{1}$-dgas, and let $f: \mathrm{T}(X) \rightarrow$ A be a map of graded algebras with cup-one products. Then $f$ commutes with the respective differentials if and only if $d_{A} \circ f(x)=f \circ d_{\top}(x)$ for all $x \in X$.

Proof. From the $\cup_{1}-d$ formula (3.7), equation (6.15), and the left Hirsch identity (3.4), it follows that if $d_{A} \circ f(x)=f \circ d_{\top}(x)$ for all $x \in X$, then $d_{A} \circ f(a)=f \circ d_{\top}(a)$ for all $a \in \mathrm{~T}^{1}(X)$. The result now follows, since $\mathrm{T}^{n}$, for $n \geq 2$, is generated by products of elements in $\mathrm{T}^{1}$, and both $d_{\mathrm{T}}$ and $d_{A}$ satisfy the graded Leibniz rule.

The next corollary follows at once from Theorem 7.5.
Corollary 7.6. Let $(\mathrm{T}(X), d)$ be the free binomial $\cup_{1}-d g a$ on a set $X$, with $d(x)=0$ for all $x \in X$. If $A$ is a binomial $\cup_{1}-d g a$, then there is a bijection between binomial $\cup_{1}-d g a$ maps from $(\mathrm{T}(X), d)$ to $\left(A, d_{A}\right)$ and set maps from $X$ to $Z^{1}(A)$.

## 8. $\mathbb{Z}_{p}$-binomial rings and binomial $\cup_{1}$-dGAS

The purpose of this section is to extend the results in the previous sections to binomial rings and $\cup_{1}$-dgas over the prime field of characteristic $p>0$.
8.1. Definition and properties of $\mathbb{Z}_{p}$-binomial algebras. Fix a prime $p$, and let $\mathbb{Z}_{p}=$ $\mathbb{Z} / p \mathbb{Z}$ be the field with $p$ elements. Let $A$ be a commutative $\mathbb{Z}_{p}$-algebra; we will assume that the structure map $\mathbb{Z}_{p} \rightarrow A$ which sends $1 \in \mathbb{Z}_{p}$ to the identity $1 \in A$ is injective. Note that the binomial operations $\zeta_{n}(a)=a(a-1) \cdots(a-n+1) / n$ ! with $a \in A$ are defined for $1 \leq n \leq p-1$, since $n!$ is then a unit in $\mathbb{Z}_{p}$.

Example 8.1. Let $A=C^{*}\left(X ; \mathbb{Z}_{p}\right)$ be the cochain algebra of a $\Delta$-complex $X$ over $\mathbb{Z}_{p}$. For a cochain $a \in A^{1}$, we have that $a(a-1) \cdots(a-n+1)=0$ for $n \geq p$, where the product is the $\cup_{1}$-product on $A^{1}$. To see this, let $e$ be any 1 -simplex in $X$; then the elements $a(e), a(e)-1, \ldots, a(e)-p+1$ are distinct elements in $\mathbb{Z}_{p}$. Since there are $p$ of these elements, one of the elements must be 0 and the property follows.

This motivates the following definition.
Definition 8.2. Let $A$ be a commutative $\mathbb{Z}_{p}$-algebra. We say that $A$ is a $\mathbb{Z}_{p}$-binomial algebra if $a(a-1) \cdots(a-n+1)=0$ for all integers $n \geq p$ and all $a \in A$.

The next step is to derive properties of binomials in a $\mathbb{Z}_{p}$-binomial algebra analogous to those for a binomial ring over $\mathbb{Z}$. We start by defining the analogue of $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$.
Given a set $X$, we will denote by $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ the quotient of the free binomial algebra $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ by the ideal generated by the elements $\zeta_{n}(x)$ for $x \in X$ and $n \geq p$, tensored with $\mathbb{Z}_{p}$. We next show that $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ has $\mathbb{Z}_{p}$-basis given by products of the elements $\zeta_{i}(x)$ for $0<i<p$ and $x \in X$. Recall from (6.10) that, for a finite subset $\mathbf{x}=\left\{x_{1}, \ldots, x_{n}\right\} \subset X$ and a finitely supported function $I: X \rightarrow \mathbb{Z}_{\geq 0}$, we write $\zeta_{I}(\mathbf{x})=\prod_{k=1}^{n} \zeta_{I\left(x_{k}\right)}\left(x_{k}\right)$.
Lemma 8.3. The ring $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is a $\mathbb{Z}_{p}$-binomial algebra, with $\mathbb{Z}_{p}$-basis given by the $\mathbb{Z}_{p^{-}}$ valued polynomials $\zeta_{I}(\mathbf{x})$ with $I: X \rightarrow\{0, \ldots, p-1\}$.

Proof. To show that $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is a $\mathbb{Z}_{p}$-binomial algebra, note that $a(a-1) \cdots(a-n+1)=$ $n!\cdot \zeta_{n}(a)$ for $a \in \operatorname{Int}\left(\mathbb{Z}^{X}\right)$. For $n \geq p$, we have that $n!$ is a multiple of $p$. Hence, for $n \geq p$ the image of $a(a-1) \cdots(a-n+1)$ is the zero element in $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$. Since the projection map $\operatorname{Int}\left(\mathbb{Z}^{X}\right) \rightarrow \operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is an epimorphism of (graded) rings, it follows that $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is a $\mathbb{Z}_{p}$-binomial algebra.

To find a basis for $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ note that from equation (6.4) it follows that $\zeta_{i}(x) \zeta_{j}(x)$ is a linear combination of elements of the form $\zeta_{\ell}(x)$ with $\max \{i, j\} \leq \ell \leq i+j$. Hence the ideal of $\operatorname{Int}\left(\mathbb{Z}^{X}\right)$ generated by the elements $\zeta_{i}(x)$ with $i \geq p$ is the $\mathbb{Z}_{p}$-subspace generated by sums of polynomials of the form $\prod_{x_{i} \in X} \zeta_{j_{i}}\left(x_{i}\right)$ with $j_{i} \geq p$ for at least one $j_{i}$. The result now follows from the integral basis theorem, Theorem 6.4.

Theorem 8.4. Let $A$ be a $\mathbb{Z}_{p}$-binomial algebra. There is then a bijection between maps of $\mathbb{Z}_{p}$-binomial algebras from $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ to $A$ and set maps from $X$ to $A$.

Proof. From the definition of $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ and Lemma 8.3, it follows that $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is the ring of finite sums of products of integer powers of the variables $x \in X$ with coefficients in $\mathbb{Z}_{p}$ without constant term, modulo the ideal generated by products of the form $a(a-$ 1) $\cdots(a-p+1)$. Hence, a map of sets $\phi: X \rightarrow A$ extends uniquely to a multiplicative map, $\tilde{\phi}: \operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right) \rightarrow A$. Since $n!$ is a unit in $\mathbb{Z}_{p}$ for $0<n<p$, it follows that $\tilde{\phi}$ commutes with the zeta maps, and the proof is complete.
Lemma 8.5. The equations (6.2) through (6.6) hold in $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$, where $\zeta_{i}(a)$ is defined only for $0<i<p$ and the binomials in (6.4) are reduced $\bmod p$.

Proof. The result follows since the projection $\operatorname{Int}\left(\mathbb{Z}^{X}\right) \rightarrow \operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$ is a map of rings that commutes with the $\zeta_{i}$ maps for $0<i<p$.

Corollary 8.6. For a $\mathbb{Z}_{p}$-binomial algebra $A$, equations (6.2) through (6.6) hold in $A$, where $\zeta_{i}(a)$ is defined only for $0<i<p$ and the binomials in (6.4) are reduced mod $p$.

Proof. This is an immediate consequence of Theorem 8.4 and Lemma 8.5.
8.2. $\mathbb{Z}_{p}$-Binomial $\cup_{1}$-dgas. We now adjust the notion introduced in Definition 6.10 to fit this context.

Definition 8.7. A differential graded algebra, $(A, d)$, over $\mathbb{Z}_{p}$ is called a $\mathbb{Z}_{p}$-binomial $\cup_{1}-d g a$ if the following conditions are satisfied.
(1) $A$ is a graded $\mathbb{Z}_{p}$-algebra with cup-one products.
(2) The differential $d$ satisfies the $\cup_{1}-d$ formula (3.7).
(3) The $\mathbb{Z}_{p}$-vector subspace $\mathbb{Z}_{p} \oplus A^{1} \subset A^{\leq 1}$, with multiplication on $A^{1}$ given by the cup-one product, is a $\mathbb{Z}_{p}$-binomial algebra.

Example 8.8. Let $X$ be a $\Delta$-complex. Using the result in Example 8.1, it is readily verified that the cochain algebra $\mathbb{Z}_{p} \oplus C^{\geq 1}\left(X ; \mathbb{Z}_{p}\right)$ is a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, where the $\zeta$ maps, $\zeta_{n}^{X}: C^{1}\left(X ; \mathbb{Z}_{p}\right) \rightarrow C^{1}\left(X ; \mathbb{Z}_{p}\right)$, are defined by setting

$$
\zeta_{n}^{X}(f)(e):=\frac{f(e)(f(e)-1) \cdots(f(e)-n+1)}{n!}
$$

for each integer $1 \leq n \leq p-1$, for each 1-cochain $f \in C^{1}\left(X ; \mathbb{Z}_{p}\right)=\operatorname{Hom}\left(C_{1}\left(X ; \mathbb{Z}_{p}\right), \mathbb{Z}_{p}\right)$, and for each 1-simplex $e$ in $X$.

Theorem 8.9. Let A be a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga. Then, for each $a \in Z^{1}(A)$ and each integer $k$ with $1 \leq k \leq p-1$, we have

$$
\begin{equation*}
d \zeta_{k}(a)=-\sum_{\ell=1}^{k-1} \zeta_{\ell}(a) \cup \zeta_{k-\ell}(a) \tag{8.1}
\end{equation*}
$$

More generally, if $I=\left(k_{1}, \ldots, k_{n}\right)$ with $1 \leq k_{i} \leq p-1$ and $\mathbf{a}=\left(a_{1}, \ldots, a_{n}\right)$ with $a_{i} \in Z^{1}(A)$, then

$$
\begin{equation*}
d\left(\zeta_{I}(\mathbf{a})\right)=-\sum_{\substack{I_{1}+I_{2}=I \\ I_{j} \neq \boldsymbol{0}}} \zeta_{I_{1}}(\mathbf{a}) \cup \zeta_{I_{2}}(\mathbf{a}) \tag{8.2}
\end{equation*}
$$

Proof. The proof follows the same steps as in the proof of Theorem 6.13 for $1 \leq k \leq$ $p-1$.

Remark 8.10. For $p=2$, equation (8.1) is vacuously true, since $d \zeta_{1}(a)=0$ by assumption. On the other hand, equation (8.2) is true, but not tautologically so. For instance, as a consequence of formula (6.14), the identity $d\left(\zeta_{1}\left(a_{1}\right) \zeta_{1}\left(a_{2}\right)\right)=-a_{1} \cup a_{2}-a_{2} \cup a_{1}$ holds over $\mathbb{Z}_{2}$.
8.3. The free $\mathbb{Z}_{p}$-binomial graded algebra. In this section we define the free $\mathbb{Z}_{p}$-binomial graded algebra, $\mathrm{T}\left(X ; \mathbb{Z}_{p}\right)$, generated by a set $X$ and show that $\mathrm{T}\left(X ; \mathbb{Z}_{p}\right)$ has properties analogous to those of $\mathrm{T}(X)$. For the rest of this section, we will abbreviate $\otimes_{\mathbb{Z}_{p}}$ by $\otimes$.

Definition 8.11. For a set $X$ and prime $p$ the free $\mathbb{Z}_{p}$-binomial graded algebra, denoted $\mathrm{T}\left(X ; \mathbb{Z}_{p}\right)$, is the free non-commutative algebra over $\mathbb{Z}_{p}$ generated by $\operatorname{Int}\left(\mathbb{Z}_{p}^{X}\right)$.

Let $\mathrm{T}=\mathrm{T}\left(X ; \mathbb{Z}_{p}\right)$. The $\mathbb{Z}_{p}$-vector space $\mathrm{T}^{1}$ comes endowed with a cup-one product map, $\mathrm{T}^{1} \otimes \mathrm{~T}^{1} \rightarrow \mathrm{~T}^{1}, a \otimes b \mapsto a b=a \cup_{1} b$. As in section 7.1, we use the cup-one product to define a map $T^{2} \otimes T^{1} \rightarrow T^{2}$ by

$$
\begin{equation*}
(a \otimes b) \otimes c \mapsto a c \otimes b+a \otimes b c \tag{8.3}
\end{equation*}
$$

For the terms in the $\cup_{1}-d$ formula to be defined, we define the map $\circ: \mathrm{T}^{2} \otimes \mathrm{~T}^{2} \rightarrow \mathrm{~T}^{2}$ by

$$
\begin{equation*}
\left(a_{1} \cup a_{2}\right) \otimes\left(b_{1} \cup b_{2}\right) \mapsto\left(a_{1} \cup_{1} b_{1}\right) \cup\left(a_{2} \cup_{1} b_{2}\right) \tag{8.4}
\end{equation*}
$$

With this structure, $\mathrm{T}\left(X ; \mathbb{Z}_{p}\right)$ is a graded $\mathbb{Z}_{p}$-algebra with cup-one products.
The following results are analogous to those in section 7. Moreover, in each case the proof follows the same steps as for the corresponding result over $\mathbb{Z}$.

Theorem 8.12. For any set $X$, the algebra $T=T\left(X ; \mathbb{Z}_{p}\right)$ is a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, with differential $d_{\mathrm{\top}}$ satisfying $d_{\mathrm{\top}}(x)=0$ for all $x \in X$.

Lemma 8.13. Let $X$ be a set, let $A$ be a $\mathbb{Z}_{p}$-binomial graded algebra, and let $\phi: X \rightarrow A^{1}$ be a map of sets. There is then a unique extension of $\phi$ to a map $f: \mathrm{T}\left(X ; \mathbb{Z}_{p}\right) \rightarrow A$ of $\mathbb{Z}_{p}$-binomial graded algebras.

Theorem 8.14. Let $\left(T\left(X ; \mathbb{Z}_{p}\right), d_{\top}\right)$ be the free $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga on a set $X$, let $\left(A, d_{A}\right)$ be a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, and let $f: \mathrm{T}\left(X ; \mathbb{Z}_{p}\right) \rightarrow A$ be a map of graded algebras over $\mathbb{Z}_{p}$ with cup-one products. Then $f$ commutes with the respective differentials if and only if $d_{A} \circ f(x)=f \circ d_{\top}(x)$ for all $x \in X$.

Corollary 8.15. Let $\left(\mathrm{T}\left(X ; \mathbb{Z}_{p}\right), d\right)$ be the free $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga on a set $X$, with $d(x)=$ 0 for all $x \in X$. If $A$ is a $\mathbb{Z}_{p}$-binomial $\cup_{1}$-dga, then there is a bijection between $\mathbb{Z}_{p^{-}}$ binomial $\cup_{1}$-dga maps from $\left(\mathrm{T}\left(X ; \mathbb{Z}_{p}\right), d\right)$ to $\left(A, d_{A}\right)$ and set maps from $X$ to $Z^{1}(A)$.

## 9. Massey products in binomial $\cup_{1}$-dgas

In this section we outline a relationship between the binomial operations and Massey products in a binomial $\cup_{1}$-dga.
9.1. Relating the $\zeta_{i}$ maps to Massey products. We start with Massey products of the form $\langle u, \ldots, u\rangle$, where $u$ is an (integral) cohomology class in degree 1 . We will develop this idea, in a more general context, in [29].

Proposition 9.1. Let A be a binomial $\cup_{1}$-dga over $\mathbb{Z}$ and let $u$ be any element in $H^{1}(A)$. For each integer $n \geq 3$, the $n$-tuple Massey product $\langle u, \ldots, u\rangle$ is defined and contains 0 .

Proof. Let $a \in A^{1}$ be a cocycle with $[a]=u$. We first treat the case $n=3$. Since $d \zeta_{2}(a)=$ $-\zeta_{1}(a) \cup \zeta_{1}(a)$, it follows that $\langle u, u, u\rangle$ is defined and contains $\left[-\zeta_{1}(a) \cup \zeta_{2}(a)-\zeta_{2}(a) \cup \zeta_{1}(a)\right]$. Since $d \zeta_{3}(a)=-\zeta_{1}(a) \cup \zeta_{2}(a)-\zeta_{2}(a) \cup \zeta_{1}(a)$, it follows that $\langle u, u, u\rangle$ contains 0 .

In the general case, it follows from equation (6.16) with $1 \leq k<n$ that $(-1)^{n} \sum_{k=1}^{n} \zeta_{k}(a) \cup$ $\zeta_{k-n}(a)$ is a cocycle with cohomology class in the $n$-tuple Massey product $\langle[a], \ldots,[a]\rangle$. Then from equation (6.16) with $k=n$ it follows that this cohomology class is zero. Thus, for any element $u \in H^{1}(A)$, the $\zeta_{k}$ maps show that $\langle u, \ldots, u\rangle$ contains 0 .

If the notion of binomial $\cup_{1}$-dga is replaced by an arbitrary $\cup_{1}$-dga, then these $n$-tuple Massey products are not necessarily defined, and if defined, may not contain 0 .

Example 9.2. Let $A$ be the subalgebra of $\mathrm{T}(\{x\})$ generated in degree one by powers of $x$. Following the steps in the proof of equation (6.16) with $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$ in place of equation (6.15), it follows that $d x^{n}=-\sum_{k=1}^{n-1}\binom{n}{k} x^{k} \cup x^{n-k}$. Therefore, $A$ is a $\cup_{1}$-dga. Moreover, $[x] \cup[x] \neq 0$ in $H^{2}(A)$, and so the Massey triple product $\langle[x],[x],[x]\rangle$ is not defined.

Thus, over the integers the Massey triple product $\langle u, u, u\rangle$ for $u \in H^{1}(A)$ may not be defined for $A$ a $\cup_{1}$-dga, but is always defined and contains zero if $A$ is a binomial $\cup_{1}$-dga.
9.2. Massey triple products in characteristic 3. As we saw in Proposition 9.1, if $A$ is a binomial $\cup_{1}$-dga (over $\mathbb{Z}$ ) and if $u$ is any element in $H^{1}(A)$, then the Massey product $\langle u, u, u\rangle$ is defined and contains 0 . If $A$ is defined over $\mathbb{Z}_{3}$, though, such a Massey product need not vanish anymore, due to the lack of a $\zeta_{3}$ map in this context; see Remark 9.4.
In this section, we use a graphical approach (based on Figure 2) to analyze this phenomenon in more detail in the case when $A=C^{*}\left(X ; \mathbb{Z}_{3}\right)$ is the cochain algebra of a $\Delta$-complex $X$, with coefficients in $\mathbb{Z}_{3}$. In the next section, we will use a different approach to study $p$-fold Massey products in $C^{*}\left(X ; \mathbb{Z}_{p}\right)$ and generalize the next result.
Let $\delta: A^{i} \rightarrow A^{i+1}$ be the Bockstein operator associated to the coefficient sequence $0 \rightarrow$ $\mathbb{Z}_{3} \rightarrow \mathbb{Z}_{9} \rightarrow \mathbb{Z}_{3} \rightarrow 0$. The following proposition shows that the element [ $-a \cup \zeta_{2}(a)-$ $\left.\zeta_{2}(a) \cup a\right] \in\langle[a],[a],[a]\rangle$ is the negative of the mod 3 Bockstein applied to $[a]$, and hence in general, is nonzero.


Figure 2. Cochains for the Massey triple product $\langle[a],[a],[a]\rangle$ in $K\left(\mathbb{Z}_{3}, 1\right)$
Proposition 9.3. Let $X$ be a $\Delta$-complex, and let $u \in H^{1}\left(X ; \mathbb{Z}_{3}\right)$; then the Massey triple product $\langle u, u, u\rangle$ contains the element $-\delta(u)$.

Proof. Since every element in $H^{1}\left(X ; \mathbb{Z}_{3}\right)$ is represented by a map from $X$ to an EilenbergMacLane space $K\left(\mathbb{Z}_{3}, 1\right)$, it follows from equation (2.2) and the naturality of the Bockstein, that it is sufficient to prove the result for $u$ a generator of $H^{1}\left(K\left(\mathbb{Z}_{3}, 1\right) ; \mathbb{Z}_{3}\right) \cong \mathbb{Z}_{3}$. For this it is enough to show that the result holds for the 2-skeleton, $Y$, of a CW-complex structure for $K\left(\mathbb{Z}_{3}, 1\right)$.

Such a space is pictured in Figure 2, which gives a $\Delta$-complex, $Y$, for the 2-dimensional CW-complex obtained by attaching a 2 -cell to a circle by an attaching map of degree 3 . As in Example 4.3, line segments transverse to the 1-cells in a 2 -dimensional $\Delta$-complex are used to picture 1-cochains. The numbers on the arrows give the values of the cochain on the 1 -cells in the $\Delta$-complex. Note that with $\mathbb{Z}_{3}$-coefficients, the 1 -cochain $a$ indicated in the figure is a cocycle and $[a]$ is a generator of $H^{1}\left(Y ; \mathbb{Z}_{3}\right)$.
The final step is to show that the cohomology class of the cocycle $-a \cup \zeta_{2}(a)-\zeta_{2}(a) \cup a$ in Figure 2 is $-\delta([a])$. Note that the support of $-a \cup \zeta_{2}(a)-\zeta_{2}(a) \cup a$ is the 2 -simplex $[0,3,6]$, and the value of this cochain on $[0,3,6]$ is -1 . Then viewing the arrows with numbers as cochains with integer coefficients, it follows that the support of dâ is also $[0,3,6]$ and the value of $d \hat{a}$ on $[0,3,6]$ is 3 , where $\hat{a}$ denotes the integer cochain indicated by the arrows labelled $a$. This completes the proof.

Remark 9.4. Note that in the proof of Proposition 9.3, the element $u=[a]$ is a generator for $H^{1}\left(K\left(\mathbb{Z}_{3}, 1\right) ; \mathbb{Z}_{3}\right) \cong \mathbb{Z}_{3}$ and $u \cup u=0$, since $d \zeta_{2}(a)=-a \cup a$. Recall from $\S 2.3$ that the indeterminacy of a Massey triple product $\left\langle u_{1}, u_{2}, u_{3}\right\rangle$ of cohomology classes in $H^{1}(A)$, where $A$ is a dga, is the set of elements in $u_{1} \cup H^{1}(A)+H^{1}(A) \cup u_{3}$. In this example, where $A=C^{*}\left(K\left(\mathbb{Z}_{3}, 1\right), \mathbb{Z}_{3}\right)$, the cup product map $H^{1} \otimes H^{1} \rightarrow H^{2}$ is the zero map, so the indeterminacy of $\langle u, u, u\rangle$ is zero. It then follows from Proposition 9.3 that $-\delta(u)$ is the only element in $\langle u, u, u\rangle$. Since $-\delta(u) \neq 0$ in $H^{2}\left(K\left(\mathbb{Z}_{3}, 1\right) ; \mathbb{Z}_{3}\right)$, this gives an example of a Massey product of the form $\langle u, u, u\rangle$ that does not vanish.
9.3. p-fold Massey products in characteristic $p$. The next result generalizes Proposition 9.3, from triple Massey products in characteristic 3 to $p$-fold Massey products in characteristic an odd prime $p$. For a $\Delta$-complex $X$, we let $\delta_{p}: H^{1}\left(X ; \mathbb{Z}_{p}\right) \rightarrow H^{2}\left(X ; \mathbb{Z}_{p}\right)$ be the Bockstein operator associated to the coefficient sequence $0 \rightarrow \mathbb{Z}_{p} \rightarrow \mathbb{Z}_{p^{2}} \rightarrow \mathbb{Z}_{p} \rightarrow 0$.
Theorem 9.5. For each $u \in H^{1}\left(X ; \mathbb{Z}_{p}\right)$ with $p$ an odd prime, the $p$-fold Massey product $\langle u, \ldots, u\rangle$ is defined and contains the element $-\delta_{p}(u)$.

Proof. Theorem 2 in [25] gives a general formula for Massey products of 1-dimensional cohomology classes in a finite CW-complex whose 1 -skeleton is a wedge of circles. The formula is in terms of the coefficients in the Magnus expansions of the words corresponding to the attaching maps of the 2-cells.

First let $Y$ be the 2-dimensional complex obtained by attaching a 2 -cell to a circle by an attaching map of degree $p$, and let $v \in H^{1}\left(Y ; \mathbb{Z}_{p}\right)$ be the cohomology class represented a 1-cocycle $a \in Z^{1}\left(Y ; \mathbb{Z}_{p}\right)$ with $a(z)=1$, where $z$ denotes the 1-chain corresponding to the circle in the 1 -skeleton. It follows from [25, Theorem 2] that the $p$-fold Massey product $\langle v, \ldots, v\rangle$ is defined and is equal to $-c_{p}(x)[b]$, where $c_{p}(x)$ is the coefficient of $x^{p} \bmod p$ in $(1+x)^{p}=1+x^{p}$, and $b \in Z^{2}\left(Y ; \mathbb{Z}_{p}\right)$ is the reduction $\bmod p$ of an integer 2-cocycle $\beta$ determined by $d \alpha=p \beta$, where $\alpha \in C^{1}(Y ; \mathbb{Z})$ is an integer cochain with $\bmod p$ reduction equal to $a$. Note that in our case $c_{p}(x)=1$, and from the equation $d \alpha=p \beta$, it follows from the definition of the Bockstein operator that $[b]=\delta_{p}([a])$. So in $Y$, we have that $\langle v, \ldots, v\rangle=-\delta_{p}(v)$.
Now, since $Y$ is the 2 -skeleton of a $K\left(\mathbb{Z}_{p}, 1\right)$, and since every element $u \in H^{1}\left(X ; \mathbb{Z}_{p}\right)$ is equal to the pullback $f^{*}(v)$ of the class $v \in H^{1}\left(K\left(\mathbb{Z}_{p}, 1\right) ; \mathbb{Z}_{p}\right)=H^{1}\left(Y ; \mathbb{Z}_{p}\right)$ for a (cellular) map $f: X \rightarrow K\left(\mathbb{Z}_{p}, 1\right)$, it follows from the naturality of Massey products and the Bockstein operators that the $p$-fold Massey product $\langle u, \ldots, u\rangle$ is defined and contains the element $-\delta_{p}(u) \in H^{2}\left(X ; \mathbb{Z}_{p}\right)$.

Remark 9.6. Since the Bockstein $\delta_{p}(v)$ is nonzero in $H^{2}\left(K\left(\mathbb{Z}_{p}, 1\right) ; \mathbb{Z}_{p}\right)$, the above proof provides examples where $p$-fold Massey products of elements $v \in H^{1}\left(K\left(\mathbb{Z}_{p}, 1\right) ; \mathbb{Z}_{p}\right)$ do not vanish.

Remark 9.7. Let $u \in H^{1}\left(X ; \mathbb{Z}_{p}\right)$ be a cohomology class represented by a cocycle $a \in$ $Z^{1}\left(X ; \mathbb{Z}_{p}\right)$. By equation (6.16), we have that $b=-\sum_{i=1}^{p-1} \zeta_{i}(a) \cup \zeta_{p-i}(a)$ is a cocycle with cohomology class an element in the $p$-fold Massey product $\langle[a], \ldots,[a]\rangle$.
9.4. Massey triple products with restricted indeterminacy. In this section we define the restricted Massey triple product, $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$. Restricted Massey triple products are invariants of 1-equivalence of binomial $\cup_{1}$-dgas where the coefficient ring $R$ is either $\mathbb{Z}$ or $\mathbb{Z}_{p}$, with $p$ a prime and $p \geq 3$. In section 9.5 we will use these invariants to give an example of a family of 2-dimensional CW-complexes that have isomorphic cohomology rings yet are not 1 -equivalent.

Let $A$ be a binomial $\cup_{1}$-dga over a ring $R$ as above. Given elements $u_{1}, u_{2} \in H^{1}(A)$ with $u_{1} \cup u_{2}=0$, let $a_{i}$ be cocycle representatives for $u_{i}$, and let $a_{12}$ be a 1-cochain such that $d a_{12}=a_{1} \cup a_{2}$. By Lemma 6.12, the cochain

$$
\begin{equation*}
\gamma\left(a_{1}, a_{2}, a_{1,2}\right):=a_{1} \cup a_{1,2}-\zeta_{2}\left(a_{1}\right) \cup a_{2} \tag{9.1}
\end{equation*}
$$

is a 2 -cocycle.
Definition 9.8. With notation as above, we define the restricted Massey triple product $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$ to be the subset of $H^{2}(A)$ consisting of all elements of the form $\left[\gamma\left(a_{1}, a_{2}, a_{1,2}\right)\right]$. The cochains $a_{1}, a_{2}, a_{1,2}$ are called a defining system for $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$.

The next lemma computes the indeterminacy of these restricted Massey triple products. We restrict our attention to the case when $A=C^{*}(X ; R)$ is the cochain algebra of a $\Delta$ complex $X$, with coefficients in the ring $R=\mathbb{Z}$ or $R=\mathbb{Z}_{p}$ with $p \geq 3$. We sketch a proof in this case, following a suggestion by the referee. The indeterminacy of restricted Massey products will covered in more generality in [29].

Lemma 9.9. The indeterminacy of $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$ is the set $u_{1} \cup H^{1}(A)$.
Proof. It suffices to show that the subset of $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$ obtained from any fixed choice of cocycles representing $u_{1}$ and $u_{2}$ is the entire set $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$. Let $a_{1}, a_{2}, a_{1,2}$ be a defining system for $\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}$. If one changes $a_{2}$ to $\tilde{a}_{2}=a_{2}+d b$ for some $b \in C^{0}(X)$ and $a_{1,2}$ to $\tilde{a}_{1.2}=a_{1,2}-a_{1} \cup b$, then $\gamma\left(a_{1}, a_{2}, a_{1,2}\right)$ changes by $d\left(\zeta_{2}\left(a_{1}\right) \cup b\right)$. On the other hand, if one changes $a_{1}$ to $\tilde{a}_{1}=a_{1}+d b$ and $a_{1,2}$ to $\tilde{a}_{1,2}=a_{1,2}+b \cup a_{2}$, then by using the formulas

$$
\begin{align*}
\zeta_{2}\left(a_{1}+d b\right) & =\zeta_{2}\left(a_{1}\right)+a_{1} \cup_{1} d b+\zeta_{2}(d b) \\
a_{1} \cup_{1} d b & =a_{1} \cup b-b \cup a_{1}  \tag{9.2}\\
d(c(b)) & =d b \cup b-\zeta_{2}(d b),
\end{align*}
$$

where $c(b) \in C^{0}(X)$ is defined by $c(b)(v)=\zeta_{2}(-b(v))$ for all vertices $v$ in $X$, it follows that $\gamma\left(a_{1}, a_{2}, a_{1,2}\right)$ changes by $d\left(b \cup a_{1,2}+c(b) \cup a_{2}\right)$, and the proof is complete.

Remark 9.10. It follows straight from the definition that restricted Massey triple products satisfy the following additivity formula.

$$
\begin{equation*}
\left\langle u, u, w_{1}\right\rangle_{r}+\left\langle u, u, w_{2}\right\rangle_{r} \subseteq\left\langle u, u, w_{1}+w_{2}\right\rangle_{r} . \tag{9.3}
\end{equation*}
$$

Lemma 9.11. Let $\varphi: A \rightarrow B$ be a morphism between two binomial cup-one $R$-algebras, and let $\varphi^{*}: H^{*}(A) \rightarrow H^{*}(B)$ be the induced morphism in cohomology. If $u_{1}, u_{2} \in H^{1}(A)$ are such that $u_{1} \cup u_{2}=0$, then $\varphi^{*}\left(\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}\right) \subseteq\left\langle\varphi^{*}\left(u_{1}\right), \varphi^{*}\left(u_{1}\right), \varphi^{*}\left(u_{2}\right)\right\rangle_{r}$. Moreover, if $\varphi$ is a 1-quasi-isomorphism, then the inclusion holds as equality.

Proof. The first claim follows from the naturality of cup-products and $\zeta$-maps. The second claim follows from the first one and Lemma 9.9.

The next result shows that restricted Massey triple products are an invariant of 1-equivalence for binomial cup-one algebras. Recall that for a graded algebra $H$, we denote by $D^{2} H$ the image of the cup product map $H^{1} \otimes H^{1} \rightarrow H^{2}$.
Proposition 9.12. Let $A$ and $B$ be two 1-equivalent binomial $\cup_{1}$-dgas over $R$. Then there is an isomorphism $\Phi: H^{1}(A) \oplus D^{2}(H(A)) \rightarrow H^{1}(B) \oplus D^{2}(H(B))$ which preserves degrees, commutes with cup products, and satisfies $\Phi\left(\left\langle u_{1}, u_{1}, u_{2}\right\rangle_{r}\right)=\left\langle\Phi\left(u_{1}\right), \Phi\left(u_{1}\right), \Phi\left(u_{2}\right)\right\rangle_{r}$ for all $u_{1}, u_{2} \in H^{1}(A)$ with $u_{1} \cup u_{2}=0$.

Proof. Recall from §2.2 that a 1-quasi-isomorphism $\varphi: A \rightarrow B$ induces an isomorphism $\varphi^{*}: H^{1}(A) \rightarrow H^{1}(B)$ and a monomorphim $\varphi^{*}: H^{2}(A) \rightarrow H^{2}(B)$. By naturality of cupproducts, the latter map restricts to an isomorphism $\varphi^{*}: D^{2}(H(A)) \rightarrow D^{2}(H(B))$. The claim now follows from the definition of 1-equivalence and the previous lemma.

Given an element $u \in H^{1}$, consider the following subset of $H^{2}$,

$$
\begin{equation*}
\langle u\rangle_{r}=\bigcup_{\left\{w \in H^{1}: u w=0\right\}}\langle u, u, w\rangle_{r} . \tag{9.4}
\end{equation*}
$$

Corollary 9.13. Let A and B be two 1-equivalent binomial $\cup_{1}$-dgas over $R$, and let $\Phi$ be the isomorphism from Proposition 9.12. Then $\Phi\left(\langle u\rangle_{r}\right)=\langle\Phi(u)\rangle_{r}$ for all $u \in H^{1}$.
9.5. Distinguishing homotopy types. Suppose now that $X$ and $Y$ are two finite $\Delta$ complexes such that their restricted triple Massey products cannot be matched as subsets of $H^{2}$ in the manner prescribed by Proposition 9.12 or Corollary 9.13. Those results then imply that $X$ and $Y$ are not homotopy equivalent. We illustrate this method of distinguishing homotopy types with an example.
Example 9.14. Given a non-negative integer $k$, set $X_{k}$ equal to the presentation 2-complex for the group $G_{k}$ with generators $x_{1}, x_{2}, x_{3}$ and relator $\left[x_{2}, x_{3}\right]\left[x_{1}, x_{2} x_{1}^{k} x_{2}^{-1}\right]$ as pictured in Figure 3. Choose a subdivision of the 2-cell in Figure 3 into a $\Delta$-complex, $Y_{k}$, whose cells are transverse to the line segments labelled $a_{1}, a_{2}, a_{3}$, and $a_{1,2}$.


Figure 3. The relator $\left[x_{2}, x_{3}\right]\left[x_{1}, x_{2} x_{1}^{k} x_{2}^{-1}\right]$

Then, as in Figures 1 and 2, the line segments with arrows and numbers determine elements in $C^{1}\left(Y_{k} ; \mathbb{Z}\right)$. The cochains determined by $a_{1}, a_{2}$, and $a_{3}$ are cocycles, $c_{i}$, whose cohomology classes are denoted by $u_{i}=u_{i, k}$. The cocycles $c_{i}$ evaluated on the 1 -cycles $x_{j}$, are given by $c_{i}\left(x_{j}\right)=\delta_{i, j}$. The cochain $c_{1,2}$ determined by $a_{1,2}$ satisfies $d c_{1,2}=c_{1} \cup c_{2}$. Using the computation of the cup product in Figure 1 as a guide, it follows that $u_{1} \cup u_{2}=$ $u_{1} \cup u_{3}=0$, while $u_{2} \cup u_{3}$ generates $H^{2}\left(Y_{k} ; \mathbb{Z}\right)=\mathbb{Z}$. Consequently, all the cohomology rings $H^{*}\left(X_{k} ; \mathbb{Z}\right) \cong H^{*}\left(Y_{k} ; \mathbb{Z}\right)$ are pairwise isomorphic. Furthermore, the triple Massey products $\left\langle u_{1}, u_{1}, u_{2}\right\rangle$ and $\left\langle u_{1}, u_{1}, u_{3}\right\rangle$ have indeterminacy equal to the whole of $H^{2}\left(X_{k} ; \mathbb{Z}\right)$; thus, these invariants do not distinguish the homotopy types of the spaces $X_{k}$, either.

On the other hand, we can use restricted triple Massey products to show that $X_{k}$ and $X_{\ell}$ are not of the same 2-type if $k \neq \ell$. To prove this claim, let $g_{k}$ be a generator of $H^{2}\left(X_{k} ; \mathbb{Z}\right)=\mathbb{Z}$. We first show that

$$
\begin{equation*}
\left\langle u_{1, k}\right\rangle_{r}=\left\{n k g_{k}: n \in \mathbb{Z}\right\} . \tag{9.5}
\end{equation*}
$$

Indeed, by Lemma 9.9 the indeterminacy of $\left\langle u_{1, k}, u_{1, k}, w\right\rangle_{r}$ is zero for all $w \in H^{1}\left(X_{k} ; \mathbb{Z}\right)$. Using Figure 3 it can be shown that

$$
\begin{equation*}
\left\langle u_{1, k}, u_{1, k}, u_{2, k}\right\rangle_{r}= \pm k g_{k} \tag{9.6}
\end{equation*}
$$

(with sign depending on the choice of $g_{k}$ ) and $\left\langle u_{1, k}, u_{1, k}, u_{3, k}\right\rangle_{r}=0$. We have already seen that in general $\left\langle u_{1, k}, u_{1, k}, u_{1 . k}\right\rangle$ contains zero. The claim now follows from formula (9.3).

Now suppose $X_{k}$ and $X_{\ell}$ have the same 2-type, i.e., $\pi_{1}\left(X_{k}\right) \cong \pi_{1}\left(X_{\ell}\right)$. Then, by Theorem 4.1, the cochain algebras of $X_{k}$ and $X_{\ell}$ are 1-quasi-equivalent. It then follows from Corollary 9.13 that there is an isomorphism

$$
\begin{equation*}
\Phi: H^{1}\left(X_{k} ; \mathbb{Z}\right) \oplus D^{2} H\left(X_{k} ; \mathbb{Z}\right) \xrightarrow{\cong} H^{1}\left(X_{\ell} ; \mathbb{Z}\right) \oplus D^{2} H\left(X_{\ell} ; \mathbb{Z}\right) \tag{9.7}
\end{equation*}
$$

such that $\Phi\left(\left\langle u_{1, k}\right\rangle_{r}\right)=\left\langle\Phi\left(u_{1, k}\right)\right\rangle_{r}$ where $D^{2} H\left(X_{k} ; \mathbb{Z}\right)$ denotes the submodule of $H^{2}\left(X_{k} ; \mathbb{Z}\right)$ whose elements are sums of cup products of elements in $H^{1}\left(X_{k} ; \mathbb{Z}\right)$.
Note that the set of all elements $u \in H^{1}\left(X_{k} ; \mathbb{Z}\right)$ with $u w=0$ for all $w \in H^{1}\left(X_{k} ; \mathbb{Z}\right)$ is the subspace of $H^{1}\left(X_{k} ; \mathbb{Z}\right)$ generated by $u_{1, k}$. Since $\Phi$ commutes with cup products and is an isomorphism on $H^{2}$, it follows that $\Phi\left(u_{1, k}\right)= \pm u_{1, \ell}$. Hence, $\Phi\left(\left\langle u_{1, k}\right\rangle_{r}\right)$ is either $\left\langle u_{1, \ell}\right\rangle_{r}$ or $\left\langle-u_{1, \ell}\right\rangle_{r}$. Since $\zeta_{2}(-a)-\zeta_{2}(a)=a$, we see that $\left\langle u_{1, \ell}\right\rangle_{r}=\left\langle-u_{1, \ell}\right\rangle_{r}$, and thus $\Phi\left(\left\langle u_{1, k}\right\rangle_{r}\right)=\left\langle u_{1, \ell}\right\rangle_{r}$. In view of (9.5), we conclude that $k=\ell$.

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