TOPOLOGY OF LINE ARRANGEMENTS

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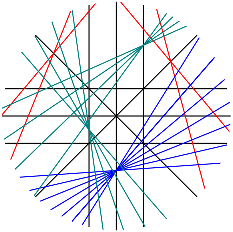


Plane algebraic curves

- Let $\mathscr{C} \subset \mathbb{CP}^2$ be a plane algebraic curve, defined by a homogeneous polynomial $f \in \mathbb{C}[z_1, z_2, z_3]$.
- In the 1930s, Zariski studied the topology of the complement, $U = \mathbb{CP}^2 \backslash \mathscr{C}$.
- He commissioned Van Kampen to find a presentation for the fundamental group, $\pi = \pi_1(U)$.
- Zariski noticed that π is *not* determined by the combinatorics of $\mathscr{C}$, but depends on the position of its singularities.
- He asked whether π is *residually finite*, i.e., whether the map to its profinite completion, $\pi \to \widehat{\pi} =: \pi^{\mathrm{alg}}$, is injective.

LINE ARRANGEMENTS

- Let $\mathscr{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial
$$f = \prod_{H\in\mathscr{A}} f_H \in \mathbb{C}[z_1, z_2, z_3],$$
with f_H linear forms so that $H = \mathbb{P}\ker(f_H)$ for each $H \in \mathscr{A}$.

- Let $L(\mathscr{A})$ be the *intersection lattice* of $\mathscr{A}$, with $L_1(\mathscr{A}) = \{\text{lines}\}$ and $L_2(\mathscr{A}) = \{\text{intersection points}\}$.

- Let $U(\mathscr{A}) = \mathbb{CP}^2 \setminus \bigcup_{H\in\mathscr{A}} H$ be the *complement* of $\mathscr{A}$.

RESIDUAL PROPERTIES OF ARRANGEMENT GROUPS

THEOREM (THOMAS KOBERDA–A.S. 2014)

Let $\mathscr{A}$ be a complexified real line arrangement, and let $\pi = \pi_1(U(\mathscr{A}))$. Then

1. *π is residually finite.*
2. *π is residually nilpotent.*
3. *π is torsion-free.*

Milnor fibration

- Let $f \in \mathbb{C}[z_1, z_2, z_3]$ be a homogeneous polynomial of degree n.
- The map $f\colon \mathbb{C}^3 \setminus \{f = 0\} \to \mathbb{C}^*$ is a smooth fibration (Milnor), with fiber $F = f^{-1}(1)$, and monodromy $h\colon F \to F$, $z \mapsto e^{2\pi i/n} z$.
- The Milnor fiber F is a regular, $\mathbb{Z}_n$-cover of $U = \mathbb{CP}^2 \setminus \{f = 0\}$.

Corollary (T.K.–A.S.)

Let $\mathscr{A}$ be an arrangement defined by a polynomial $f \in \mathbb{R}[z_1, z_2, z_3]$, let $F = F(\mathscr{A})$ be its Milnor fiber, and let $\pi = \pi_1(F)$. Then

1. *π is residually finite.*
2. *π is residually nilpotent.*
3. *π is torsion-free.*

- Let $\Delta(t) = \det(tI - h_*)$ be the characteristic polynomial of the algebraic monodromy, $h_*\colon H_1(F,\mathbb{C}) \to H_1(F,\mathbb{C})$.

PROBLEM

When f is the defining polynomial of an arrangement $\mathscr{A}$, is $\Delta = \Delta_{\mathscr{A}}$ determined solely by $L(\mathscr{A})$?

THEOREM (STEFAN PAPADIMA–A.S. 2014)

Suppose $\mathscr{A}$ has only double and triple points. Then

$$\Delta_{\mathscr{A}}(t) = (t-1)^{|\mathscr{A}|-1} \cdot (t^2+t+1)^{\beta_3(\mathscr{A})},$$

where $\beta_3(\mathscr{A})$ is an integer between 0 and 2 that depends only on $L(\mathscr{A})$.

TECHNIQUES

- Common themes:
  - Homology with coefficients in rank 1 local systems.
  - Homology of finite abelian covers.
- Specific techniques for residual properties:
  - Boundary manifold of line arrangement.
  - Towers of congruence covers.
  - The RFRp property.
- Specific techniques for Milnor fibration:
  - Nets, multinets, and pencils.
  - Cohomology jump loci (in characteristic 0 and p).
  - Modular bounds for twisted Betti numbers.

THE RFRp PROPERTY

Let G be a finitely generated group and let p be a prime.

We say that G is *residually finite rationally* p if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that

1. $G_{i+1} \lhd G_i$.
2. $\bigcap_{i\geqslant 0} G_i = \{1\}$.
3. G_i / G_{i+1} is an elementary abelian p-group.
4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.

Remarks:

- May assume each $G_i \lhd G$.
- Compare with Agol's RFRS property, where G_i / G_{i+1} only finite.
- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite and residually nilpotent.
- G RFRp $\Rightarrow$ G RFRS $\Rightarrow$ torsion-free.

- The class of RFRp groups is closed under the following operations:
  1. Taking subgroups.
  2. Finite direct products.
  3. Finite free products.
- The following groups are RFRp:
  1. Finitely generated free groups.
  2. Closed, orientable surface groups.
  3. Right-angled Artin groups.

BOUNDARY MANIFOLDS

- Let N be a regular neighborhood of $\bigcup_{H\in\mathscr{A}} H$ inside $\mathbb{CP}^2$.
- Let $\overline{U} = \mathbb{CP}^2 \setminus \operatorname{int}(N)$ be the *exterior* of $\mathscr{A}$.
- The *boundary manifold* of $\mathscr{A}$ is
$$M = \partial\overline{U} = \partial N,$$
a compact, orientable, smooth manifold of dimension 3.

EXAMPLE

Let $\mathscr{A}$ be a pencil of n hyperplanes in $\mathbb{C}^2$, defined by $f = z_1^n - z_2^n$. If $n = 1$, then $M = S^3$. If $n > 1$, then $M = \sharp^{n-1} S^1 \times S^2$.

EXAMPLE

Let $\mathscr{A}$ be a near-pencil of n planes in $\mathbb{CP}^2$, defined by $f = z_1(z_2^{n-1} - z_3^{n-1})$. Then $M = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- Work of Hirzebruch, Jiang–Yau, and E. Hironaka shows that $M = M_\Gamma$ is a graph-manifold.
- The graph Γ is the incidence graph of $\mathscr{A}$, with vertex set $V(\Gamma) = L_1(\mathscr{A}) \cup L_2(\mathscr{A})$ and edge set $E(\Gamma) = \{(H, P) \mid P \in H\}$.
- For each $v \in V(\Gamma)$, there is a vertex manifold $M_v = S^1 \times S_v$, with
$$S_v = S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w},$$
a sphere with $\deg v$ disjoint open disks removed.
- For each $e \in E(\Gamma)$, there is an edge manifold $M_e = S^1 \times S^1$.
- Vertex manifolds are glued along edge manifolds via flips.

- The inclusion $i\colon M \to U$ induces a surjection $i_\sharp\colon \pi_1(M) \twoheadrightarrow \pi_1(U)$.
- By collapsing each vertex manifold of $M = M_\Gamma$ to a point, we obtain a map $\kappa\colon M \to \Gamma$.
- Using work of D. Cohen–A.S. (2006, 2008), we get a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_1(U,\mathbb{Z}) \arrow[r] & H_1(M,\mathbb{Z}) \arrow[r, "\kappa_*"] \arrow[l, bend right, "i_*"'] & H_1(\Gamma,\mathbb{Z}) \arrow[r] \arrow[l, bend right, dotted] & 0 \end{tikzcd}.$$

LEMMA

Suppose $\mathscr{A}$ is an essential line arrangement in $\mathbb{CP}^2$. Then, for each $v \in V(\Gamma)$ and $e \in E(\Gamma)$, the inclusions $i_v\colon M_v \hookrightarrow M$ and $i_e\colon M_e \hookrightarrow M$ induce split injections on H_1, whose images are contained in $\ker(\kappa_*)$.

Using work of E. Hironaka (2001), we obtain:

LEMMA

Suppose $\mathscr{A}$ is the complexification of a real arrangement. There is then a finite, simplicial graph $\mathscr{G}$ and an embedding $j\colon \mathscr{G} \hookrightarrow M$ such that:

1. *The graph $\mathscr{G}$ is homotopy equivalent to the incidence graph Γ.*
2. *We have an exact sequence,*
$$0 \longrightarrow H_1(\mathscr{G}, \mathbb{Z}) \xrightarrow{j_*} H_1(M, \mathbb{Z}) \xrightarrow{i_*} H_1(U, \mathbb{Z}) \longrightarrow 0\ .$$
3. *We have an exact sequence,*
$$1 \longrightarrow \pi_1(\mathscr{G}) \xrightarrow{j_\sharp} \pi_1(M) \xrightarrow{i_\sharp} \pi_1(U) \longrightarrow 1\ .$$

TOWERS OF CONGRUENCE COVERS

- For each prime p, we construct a tower of regular covers of M,

$$\cdots \longrightarrow M_{i+1} \xrightarrow{q_{i+1}} M_i \xrightarrow{q_i} \cdots \xrightarrow{q_1} M_0 = M.$$

- Each M_i is a graph-manifold, modelled on a graph Γ_i.
- The group of deck-transformations for q_{i+1} is the elementary abelian p-group $\big((H_1(M_i,\mathbb{Z})/\mathrm{tors})/H_1(\Gamma_i,\mathbb{Z})\big)\otimes\mathbb{Z}_p$.
- The covering maps preserve the graph-manifold structures, e.g.,

$$\begin{tikzcd} M_{v,i} \arrow[r] \arrow[d, "q_v"] & M_i \arrow[d, "q"] \\ M_v \arrow[r] & M \end{tikzcd}$$

where $M_{v,i}$ is a connected component of $q^{-1}(M_v)$ and $q_v = q|_{M_{v,i}}$.

- The inclusions $M_{v,i} \hookrightarrow M_i$ and $M_{e,i} \hookrightarrow M_i$ induce injections on H_1, whose images are contained in $\ker((\kappa_i)_*)$.
- If $\mathscr{A}$ is complexified real, the graph $\mathscr{G} \hookrightarrow M$ lifts to a graph $\mathscr{G}_i \hookrightarrow M_i$ so that
  - The group $H_1(M_i, \mathbb{Z})$ splits off $H_1(\mathscr{G}_i, \mathbb{Z})$ as a direct summand.
  - $H_1(\mathscr{G}_i, \mathbb{Z}) \cap H_1(M_{v,i}, \mathbb{Z}) = 0$, for all $v \in V(\Gamma)$.

Finally,

- For each $v \in V(\Gamma)$, the group $\pi_1(M_v) = \mathbb{Z} \times \pi_1(S_v)$ is RFRp.
- From the construction of the tower, it follows that $\pi_1(M)$ is RFRp.
- If $\mathscr{A}$ is complexified real, the above properties of the lifts of $\mathscr{G}$ imply that $\pi_1(U) = \pi_1(M)/\langle\langle j_\sharp(\pi_1(\mathscr{G}))\rangle\rangle$ is also RFRp.

RESONANCE VARIETIES AND MULTINETS

- Let $X(\mathscr{A}) = \mathbb{C}^3 \setminus \bigcup_{H \in \mathscr{A}} \ker(f_H)$, so that $U(\mathscr{A}) = \mathbb{P}X(\mathscr{A})$ and $X(\mathscr{A}) \cong \mathbb{C}^* \times U(\mathscr{A})$.
- Let $A = H^*(X(\mathscr{A}), \Bbbk)$: an algebra that depends only on $L(\mathscr{A})$ and the field $\Bbbk$ (Orlik and Solomon).
- For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$
- The (degree 1) *resonance varieties* of $\mathscr{A}$ are the cohomology jump loci of this "Aomoto complex":

$$\mathscr{R}_s(\mathscr{A}, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^1(A, \cdot a) \geqslant s\},$$

Work of Arapura, Falk, Cohen–A.S., Libgober–Yuzvinsky, and Falk–Yuzvinsky completely describes the varieties $\mathscr{R}_s(\mathscr{A}, \mathbb{C})$:

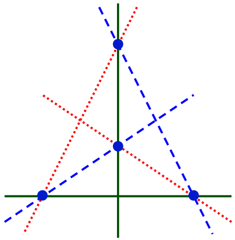
- $\mathscr{R}_1(\mathscr{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(X(\mathscr{A}), \mathbb{C}) \cong \mathbb{C}^{|\mathscr{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathscr{R}_s(\mathscr{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s+1$.
- Each k*-multinet* on a sub-arrangement $\mathscr{B} \subseteq \mathscr{A}$ gives rise to a component of $\mathscr{R}_1(\mathscr{A}, \mathbb{C})$ of dimension $k-1$. Moreover, all components of $\mathscr{R}_1(\mathscr{A}, \mathbb{C})$ arise in this way.

DEFINITION (FALK AND YUZVINSKY)

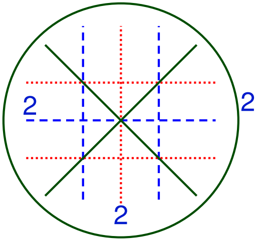
A *multinet* on $\mathscr{A}$ is a partition of the set $\mathscr{A}$ into $k \geqslant 3$ subsets $\mathscr{A}_1, \dots, \mathscr{A}_k$, together with an assignment of multiplicities, $m\colon \mathscr{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathscr{A})$, called the base locus, such that:

1. There is an integer d such that $\sum_{H\in\mathscr{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H\in\mathscr{A}_\alpha : H\supset X} m_H$ is independent of α.
4. Each set $\left(\bigcup_{H\in\mathscr{A}_\alpha} H\right)\backslash\mathcal{X}$ is connected.

- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathscr{A}$.
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.



A $(3,2)$-net on the A_3 arrangement
$\mathcal{X}$ consists of 4 triple points ($n_X = 1$)



A $(3,4)$-multinet on the B_3 arrangement
$\mathcal{X}$ consists of 4 triple points ($n_X = 1$)
and 3 triple points ($n_X = 2$)

- (Yuzvinsky and Pereira–Yuz): If $\mathscr{A}$ supports a k-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; if the multinet is not reduced, then $k = 3$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4,3)$-net.
- (Cordovil–Forge and Torielli–Yoshinaga): There are no 4-nets on real arrangements.

Modular inequalities

- Recall $\Delta(t)$ is the characteristic polynomial of the algebraic monodromy of the Milnor fibration, $h_* \colon H_1(F, \mathbb{C}) \to H_1(F, \mathbb{C})$.
- Set $n = |\mathscr{A}|$. Since $h^n = \mathrm{id}$, we have

$$\Delta(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathscr{A})},$$

  where $\Phi_d(t)$ is the d-th cyclotomic polynomial, and $e_d(\mathscr{A}) \in \mathbb{Z}_{\geqslant 0}$.
- If there is a non-transverse multiple point on $\mathscr{A}$ of multiplicity not divisible by d, then $e_d(\mathscr{A}) = 0$ (Libgober 2002).
- In particular, if $\mathscr{A}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $\sigma = \sum_{H \in \mathscr{A}} e_H \in A^1$ be the "diagonal" vector.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathscr{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathscr{A}) = \max\{s \mid \sigma \in \mathscr{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–A.S. 2010)

$e_{p^s}(\mathscr{A}) \leqslant \beta_p(\mathscr{A})$, for all $s \geqslant 1$.

THEOREM (S.P.–A.S.)

1. *Suppose $\mathscr{A}$ admits a k-net. Then $\beta_p(\mathscr{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathscr{A}) \geqslant k - 2$, otherwise.*
2. *If $\mathscr{A}$ admits a reduced k-multinet, then $e_k(\mathscr{A}) \geqslant k - 2$.*

COMBINATORICS AND MONODROMY

THEOREM (S.P.–A.S.)

Suppose $\mathscr{A}$ has no points of multiplicity $3r$ with $r > 1$. Then, the following conditions are equivalent:

1. $\mathscr{A}$ *admits a reduced* 3*-multinet.*
2. $\mathscr{A}$ *admits a* 3*-net.*
3. $\beta_3(\mathscr{A}) \neq 0$.

Moreover, the following hold:

4. $\beta_3(\mathscr{A}) \leqslant 2$.
5. $e_3(\mathscr{A}) = \beta_3(\mathscr{A})$*, and thus* $e_3(\mathscr{A})$ *is combinatorially determined.*

THEOREM (S.P.–A.S.)

Suppose $\mathscr{A}$ supports a 4-net and $\beta_2(\mathscr{A}) \leqslant 2$. Then

$$e_2(\mathscr{A}) = e_4(\mathscr{A}) = \beta_2(\mathscr{A}) = 2.$$

CONJECTURE (S.P.–A.S.)

Let $\mathscr{A}$ be an arrangement which is not a pencil. Then

$$e_{p^s}(\mathscr{A}) = 0$$

for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathscr{A}) = e_4(\mathscr{A}) = \beta_2(\mathscr{A}) \text{ and } e_3(\mathscr{A}) = \beta_3(\mathscr{A}).$$

If $e_d(\mathscr{A}) = 0$ for all divisors d of $|\mathscr{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathscr{A}}(t) = (t-1)^{|\mathscr{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathscr{A})}(t^2+t+1)^{\beta_3(\mathscr{A})}.$$

The conjecture has been verified for several classes of arrangements, including complex reflection arrangements and certain types of real arrangements.