Resonance, representations, and the Johnson filtration

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FILTRATIONS AND GRADED LIE ALGEBRAS

Let π be a group, with commutator $(x, y) = xyx^{-1}y^{-1}$.
Suppose given a descending filtration

$$\pi = \Phi^1 \supseteq \Phi^2 \supseteq \cdots \supseteq \Phi^s \supseteq \cdots$$

by subgroups of π, satisfying

$$(\Phi^s, \Phi^t) \subseteq \Phi^{s+t}, \quad \forall s, t \geqslant 1.$$

Then $\Phi^s \lhd \pi$, and Φ^s/Φ^{s+1} is abelian. Set

$$\operatorname{gr}_\Phi(\pi) = \bigoplus_{s \geqslant 1} \Phi^s/\Phi^{s+1}.$$

This is a graded Lie algebra, with bracket $[\ ,\ ]\colon \operatorname{gr}^s_\Phi \times \operatorname{gr}^t_\Phi \to \operatorname{gr}^{s+t}_\Phi$ induced by the group commutator.

Basic example: the *lower central series*, $\Gamma^s = \Gamma^s(\pi)$, defined as

$$\Gamma^1 = \pi,\ \Gamma^2 = \pi',\ \dots,\ \Gamma^{s+1} = (\Gamma^s, \pi),\ \dots$$

Then for any filtration Φ as above, $\Gamma^s \subseteq \Phi^s$. Thus, we have a morphism of graded Lie algebras,

$$\iota_\Phi \colon \operatorname{gr}_\Gamma(\pi) \longrightarrow \operatorname{gr}_\Phi(\pi)\ .$$

EXAMPLE (P. HALL, E. WITT, W. MAGNUS)

Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n. Then:

- F_n is residually nilpotent, i.e., $\bigcap_{s\geqslant 1} \Gamma^s(F_n) = \{1\}$.
- $\operatorname{gr}_\Gamma(F_n)$ is isomorphic to the free Lie algebra $\mathcal{L}_n = \operatorname{Lie}(\mathbb{Z}^n)$.
- $\operatorname{gr}^s_\Gamma(F_n)$ is free abelian, of rank $\frac{1}{s}\sum_{d|s} \mu(d) n^{\frac{s}{d}}$.
- If $n \geqslant 2$, the center of $\mathcal{L}_n$ is trivial.

AUTOMORPHISM GROUPS

Let $\mathrm{Aut}(\pi)$ be the group of all automorphisms $\alpha\colon \pi \to \pi$, with $\alpha \cdot \beta := \alpha \circ \beta$. The *Andreadakis–Johnson filtration*,

$$\mathrm{Aut}(\pi) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^s \supseteq \cdots$$

has terms $F^s = F^s(\mathrm{Aut}(\pi))$ consisting of those automorphisms which act as the identity on the s-th nilpotent quotient of π:

$$\begin{aligned} F^s &= \ker\big(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\Gamma^{s+1})\big) \\ &= \{\alpha \in \mathrm{Aut}(\pi) \mid \alpha(x)\cdot x^{-1} \in \Gamma^{s+1},\ \forall x \in \pi\} \end{aligned}$$

Kaloujnine [1950]: $(F^s, F^t) \subseteq F^{s+t}$.
First term is the *Torelli group*,

$$\mathcal{I}_\pi = F^1 = \ker\big(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi_{\mathrm{ab}})\big).$$

By construction, $F^1 = \mathcal{I}_G$ is a normal subgroup of $F^0 = \mathrm{Aut}(\pi)$. The quotient group,

$$\mathcal{A}(\pi) = F^0/F^1 = \mathrm{im}(\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi_{\mathrm{ab}}))$$

is the *symmetry group* of $\mathcal{I}_\pi$; it fits into the exact sequence

$$1 \longrightarrow \mathcal{I}_\pi \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \mathcal{A}(\pi) \longrightarrow 1 .$$

The Torelli group comes endowed with two filtrations:

- The Johnson filtration $\{F^s(\mathcal{I}_\pi)\}_{s\geqslant 1}$, inherited from $\mathrm{Aut}(\pi)$.
- The lower central series filtration, $\{\Gamma^s(\mathcal{I}_\pi)\}$.

The respective associated graded Lie algebras, $\mathrm{gr}_F(\mathcal{I}_\pi)$ and $\mathrm{gr}_\Gamma(\mathcal{I}_\pi)$, come endowed with natural actions of $\mathcal{A}(\pi)$; moreover, the morphism $\iota_F \colon \mathrm{gr}_\Gamma(\mathcal{I}_\pi) \to \mathrm{gr}_F(\mathcal{I}_\pi)$ is $\mathcal{A}(\pi)$-equivariant.

THE JOHNSON HOMOMORPHISM

Given a graded Lie algebra $\mathfrak{g}$, let

$$\mathrm{Der}^s(\mathfrak{g}) = \{\delta\colon \mathfrak{g}^\bullet \to \mathfrak{g}^{\bullet+s} \text{ linear} \mid \delta[x,y] = [\delta x, y] + [x, \delta y], \forall x, y \in \mathfrak{g}\}.$$

Then $\mathrm{Der}(\mathfrak{g}) = \bigoplus_{s\geqslant 1} \mathrm{Der}^s(\mathfrak{g})$ is a graded Lie algebra, with bracket $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$.

THEOREM

Given a group π, there is a monomorphism of graded Lie algebras,

$$J\colon \mathrm{gr}_F(\mathcal{I}_\pi) \longrightarrow \mathrm{Der}(\mathrm{gr}_\Gamma(\pi)) ,$$

given on homogeneous elements $\alpha \in F^s(\mathcal{I}_\pi)$ and $x \in \Gamma^t(\pi)$ by

$$J(\bar\alpha)(\bar x) = \overline{\alpha(x) \cdot x^{-1}}.$$

Moreover, J is equivariant with respect to the natural actions of $\mathcal{A}(\pi)$.

The Johnson homomorphism informs on the Johnson filtration.

THEOREM

Suppose $Z(\mathrm{gr}_\Gamma(\pi)) = 0$. *For each* $q \geqslant 1$, *the following are equivalent:*

1. $J \circ \iota_F \colon \mathrm{gr}^s_\Gamma(\mathcal{I}_\pi) \to \mathrm{Der}^s(\mathrm{gr}_\Gamma(\pi))$ *is injective, for all* $s \leqslant q$.
2. $\Gamma^s(\mathcal{I}_\pi) = F^s(\mathcal{I}_\pi)$, *for all* $s \leqslant q+1$.

In particular, if $\mathrm{gr}_\Gamma(\pi)$ is centerless and $J \circ \iota_F \colon \mathrm{gr}^1_\Gamma(\mathcal{I}_\pi) \to \mathrm{Der}^1(\mathrm{gr}_\Gamma(\pi))$ is injective, then $F^2(\mathcal{I}_\pi) = \mathcal{I}'_\pi$.

PROBLEM

Determine the homological finiteness properties of the groups $F^s(\mathcal{I}_\pi)$. *In particular, decide whether* $\dim H_1(\mathcal{I}'_\pi, \mathbb{Q}) < \infty$.

AN OUTER VERSION

Let $\mathrm{Inn}(\pi) = \mathrm{im}(\mathrm{Ad}\colon \pi \to \mathrm{Aut}(\pi))$, where $\mathrm{Ad}_x\colon \pi \to \pi$, $y \mapsto xyx^{-1}$.
Define the *outer* automorphism group of π by

$$1 \longrightarrow \mathrm{Inn}(\pi) \longrightarrow \mathrm{Aut}(\pi) \xrightarrow{q} \mathrm{Out}(\pi) \longrightarrow 1 \,.$$

We then have

- Filtration $\{\widetilde{F}^s\}_{s\geqslant 0}$ on $\mathrm{Out}(\pi)$: $\widetilde{F}^s := q(F^s)$.
- The *outer Torelli group* of π: subgroup $\widetilde{\mathcal{I}}_\pi = \widetilde{F}^1$ of $\mathrm{Out}(\pi)$.
- Exact sequence: $1 \longrightarrow \widetilde{\mathcal{I}}_\pi \longrightarrow \mathrm{Out}(\pi) \longrightarrow \mathcal{A}(\pi) \longrightarrow 1 \,.$

THEOREM

Suppose $Z(\mathrm{gr}_\Gamma(\pi)) = 0$. Then the Johnson homomorphism induces an $\mathcal{A}(\pi)$-equivariant monomorphism of graded Lie algebras,

$$\widetilde{J}\colon\ \mathrm{gr}_{\widetilde{F}}(\widetilde{\mathcal{I}}_\pi) \longrightarrow \widetilde{\mathrm{Der}}(\mathrm{gr}_\Gamma(\pi)) \,,$$

where $\widetilde{\mathrm{Der}}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) / \mathrm{im}(\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}))$.

THE ALEXANDER INVARIANT

- Let π be a group, and $\pi_{\text{ab}} = \pi/\pi'$ its maximal abelian quotient.
- Let $\pi'' = (\pi', \pi')$; then π/π'' is the maximal metabelian quotient. Get exact sequence $0 \longrightarrow \pi'/\pi'' \longrightarrow \pi/\pi'' \longrightarrow \pi_{\text{ab}} \longrightarrow 0$.
- Conjugation in π/π'' turns the abelian group

$$B(\pi) := \pi'/\pi'' = H_1(\pi', \mathbb{Z})$$

into a module over $R = \mathbb{Z}\pi_{\text{ab}}$, called the *Alexander invariant* of π.
- Since both π' and π'' are characteristic subgroups of π, the action of $\text{Aut}(\pi)$ on π induces an action on $B(\pi)$. This action need not respect the R-module structure. Nevertheless:

PROPOSITION

The Torelli group $\mathcal{I}_\pi$ acts R-linearly on the Alexander invariant $B(\pi)$.

CHARACTERISTIC VARIETIES

- Assume now that π is finitely generated.
- Let $\widehat{\pi} = \mathrm{Hom}(\pi, \mathbb{C}^*)$ be its *character group*: an algebraic group, with coordinate ring $\mathbb{C}[\pi_{\mathrm{ab}}]$.
- The map $\mathrm{ab}\colon \pi \twoheadrightarrow \pi_{\mathrm{ab}}$ induces an isomorphism $\widehat{\pi_{\mathrm{ab}}} \xrightarrow{\simeq} \widehat{\pi}$.
- $\widehat{\pi}^{\circ} \cong (\mathbb{C}^*)^n$, where $n = \operatorname{rank} \pi_{\mathrm{ab}}$.

DEFINITION

The (first) *characteristic variety* of π is the support of the (complexified) Alexander invariant $B = B(\pi) \otimes \mathbb{C}$:

$$\mathcal{V}(\pi) := V(\operatorname{ann} B) \subset \widehat{\pi}.$$

This variety informs on the Betti numbers of normal subgroups $N \triangleleft \pi$ with π/N abelian. In particular (for $N = \pi'$):

PROPOSITION

The set $\mathcal{V}(\pi)$ is finite if and only if $b_1(\pi') = \dim_{\mathbb{C}} B(\pi) \otimes \mathbb{C}$ is finite.

RESONANCE VARIETIES

Let V be a finite-dimensional $\mathbb{C}$-vector space, and let $K \subset V \wedge V$ be a subspace.

DEFINITION

The *resonance variety* $\mathcal{R} = \mathcal{R}(V, K)$ is the set of elements $a \in V^*$ for which there is an element $b \in V^*$, not proportional to a, such that $a \wedge b$ belongs to the orthogonal complement $K^{\perp} \subseteq V^* \wedge V^* = (V \wedge V)^*$.

- $\mathcal{R}$ is a conical, Zariski-closed subset of the affine space V^*.
- For instance, if $K = 0$ and $\dim V > 1$, then $\mathcal{R} = V^*$.
- At the other extreme, if $K = V \wedge V$, then $\mathcal{R} = 0$.

The resonance variety $\mathcal{R}$ has several other interpretations.

KOSZUL MODULES

- Let $S = \mathrm{Sym}(V)$ be the symmetric algebra on V.
- Let $(S \otimes_{\mathbb{C}} \bigwedge V, \delta)$ be the Koszul resolution, with differential $\delta_p \colon S \otimes_{\mathbb{C}} \bigwedge^p V \to S \otimes_{\mathbb{C}} \bigwedge^{p-1} V$ given by

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \sum_{j=1}^{p} (-1)^{j-1} v_{i_j} \otimes (v_{i_1} \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_{i_p}).$$

- Let $\iota \colon K \to V \wedge V$ be the inclusion map.
- The *Koszul module* $\mathcal{B}(V, K)$ is the graded S-module presented as

$$S \otimes_{\mathbb{C}} \left(\bigwedge^3 V \oplus K\right) \xrightarrow{\delta_3 + \mathrm{id} \otimes \iota} S \otimes_{\mathbb{C}} \bigwedge^2 V \twoheadrightarrow \mathcal{B}(V, K) .$$

PROPOSITION

The resonance variety $\mathcal{R} = \mathcal{R}(V, K)$ is the support of the Koszul module $\mathcal{B} = \mathcal{B}(V, K)$:

$$\mathcal{R} = V(\mathrm{ann}(\mathcal{B})) \subset V^*.$$

In particular, $\mathcal{R} = 0$ if and only if $\dim_{\mathbb{C}} \mathcal{B} < \infty$.

COHOMOLOGY JUMP LOCI

- Let $A = A(V, K)$ be the quadratic algebra defined as the quotient of the exterior algebra $E = \bigwedge V^*$ by the ideal generated by $K^\perp \subset V^* \wedge V^* = E^2$.
- Then $\mathcal{R}$ is the set of points $a \in A^1$ where the cochain complex

$$A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2$$

is not exact (in the middle).

- The graded pieces of the (dual) Koszul module can be reinterpreted in terms of the linear strand in a Tor module:

$$\mathcal{B}_q^* \cong \operatorname{Tor}^E_{q+1}(A, \mathbb{C})_{q+2}$$

VANISHING RESONANCE

Setting $m = \dim K$, we may view K as a point in $\mathrm{Gr}_m(V \wedge V)$, and $\mathbb{P}(K^\perp)$ as a codimension m projective subspace in $\mathbb{P}(V^* \wedge V^*)$.

LEMMA

Let $\mathrm{Gr}_2(V^*) \hookrightarrow \mathbb{P}(V^* \wedge V^*)$ *be the Plücker embedding. Then,*

$$\mathcal{R}(V, K) = 0 \Longleftrightarrow \mathbb{P}(K^\perp) \cap \mathrm{Gr}_2(V^*) = \emptyset.$$

THEOREM

For any integer m *with* $0 \leqslant m \leqslant \binom{n}{2}$, *where* $n = \dim V$, *the set*

$$U_{n,m} = \{K \in \mathrm{Gr}_m(V \wedge V) \mid \mathcal{R}(V, K) = 0\}$$

is Zariski open. Moreover, this set is non-empty if and only if $m \geqslant 2n - 3$, *in which case there is an integer* $q = q(n, m)$ *such that* $\mathcal{B}_q(V, K) = 0$, *for every* $K \in U_{n,m}$.

RESONANCE VARIETIES OF GROUPS

- The resonance variety of a f.g. group π is defined as

$$\mathcal{R}(\pi) = \mathcal{R}(V, K),$$

where $V^* = H^1(\pi, \mathbb{C})$ and $K^\perp = \ker(\cup_\pi \colon V^* \wedge V^* \to H^2(\pi, \mathbb{C}))$.

- Rationally, every resonance variety arises in this fashion:

PROPOSITION

Let V be a finite-dimensional $\mathbb{C}$-vector space, and let $K \subseteq V \wedge V$ be a linear subspace, defined over $\mathbb{Q}$. Then, there is a finitely presented, commutator-relators group π with $V^* = H^1(\pi, \mathbb{C})$ and $K^\perp = \ker(\cup_\pi)$.

- $\mathcal{R} = \mathcal{R}(\pi)$ is an approximation to $\mathcal{V} = \mathcal{V}(\pi)$.

THEOREM (LIBGOBER, DIMCA–PAPADIMA–S.)

Let $\mathrm{TC}_1(\mathcal{V})$ be the tangent cone to $\mathcal{V}$ at 1, viewed as a subset of $T_1(\hat{\pi}) = H^1(\pi, \mathbb{C})$. Then $\mathrm{TC}_1(\mathcal{V}) \subseteq \mathcal{R}$. Moreover, if π is 1-formal, then equality holds, and $\mathcal{R}$ is a union of rational subspaces.

EXAMPLE (RIGHT-ANGLED ARTIN GROUPS)

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a (finite, simple) graph. The corresponding *right-angled Artin group* is

$$\pi_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle.$$

- $V = H_1(\pi_\Gamma, \mathbb{C})$ is the vector space spanned by V.
- $K \subseteq V \wedge V$ is spanned by $\{v \wedge w \mid \{v, w\} \in \mathsf{E}\}$.
- $A = A(V, K)$ is the exterior Stanley–Reisner ring of Γ.
- $\mathcal{R}(\pi_\Gamma)$ is the union of all coordinate subspaces $\mathbb{C}^{\mathsf{W}} \subset \mathbb{C}^{\mathsf{V}}$, taken over all $\mathsf{W} \subset \mathsf{V}$ for which the induced graph Γ_{W} is disconnected.
- $\sum_{q \geqslant 0} \dim_{\mathbb{C}}(\mathcal{B}_q) t^{q+2} = Q_\Gamma(t/(1-t))$, where
$$Q_\Gamma(t) = \sum_{k \geqslant 0} \sum_{\mathsf{W} \subset \mathsf{V}:\ |\mathsf{W}| = k} \tilde{b}_0(\Gamma_{\mathsf{W}}) t^k.$$

ROOTS, WEIGHTS, AND VANISHING RESONANCE

- Let $\mathfrak{g}$ be a complex, semisimple Lie algebra.
- Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of simple roots $\Delta \subset \mathfrak{h}^*$.
- Let $(\ ,\ )$ be the inner product on $\mathfrak{h}^*$ defined by the Killing form.
- Each simple root $\beta \in \Delta$ gives rise to elements $x_\beta, y_\beta \in \mathfrak{g}$ and $h_\beta \in \mathfrak{h}$ which generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.
- Each irreducible representation of $\mathfrak{g}$ is of the form $V(\lambda)$, where $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ is a dominant weight.
- A non-zero vector $v \in V(\lambda)$ is a maximal vector (of weight λ) if $x_\beta \cdot v = 0$, for all $\beta \in \Delta$. Such a vector is uniquely determined (up to non-zero scalars), and is denoted by v_λ.

LEMMA

The representation $V(\lambda) \wedge V(\lambda)$ contains a direct summand isomorphic to $V(2\lambda - \beta)$, for some simple root β, if and only if $(\lambda, \beta) \neq 0$. When it exists, such a summand is unique.

THEOREM

Let $V = V(\lambda)$ be an irreducible $\mathfrak{g}$-module, and let $K \subset V \wedge V$ be a submodule. Let $V^* = V(\lambda^*)$ be the dual module, and let v_{λ^*} be a maximal vector for V^*.

1. *Suppose there is a root $\beta \in \Delta$ such that $(\lambda^*, \beta) \neq 0$, and suppose the vector $v_{\lambda^*} \wedge y_\beta v_{\lambda^*}$ (of weight $2\lambda^* - \beta$) belongs to $K^\perp$. Then $\mathcal{R}(V, K) \neq 0$.*
2. *Suppose that $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root β. Then $\mathcal{R}(V, K) = 0$.*

COROLLARY

$\mathcal{R}(V, K) = 0$ if and only if $2\lambda^* - \beta$ is not a dominant weight for $K^\perp$, for any simple root β such that $(\lambda^*, \beta) \neq 0$.

THE CASE OF $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

- $\mathfrak{h}^*$ is spanned t_1 and t_2 (the dual coordinates on the subspace of diagonal 2×2 complex matrices), subject to $t_1 + t_2 = 0$.
- There is a single simple root, $\beta = t_1 - t_2$.
- The defining representation is $V(\lambda_1)$, where $\lambda_1 = t_1$.
- The irreps are of the form

$$V_n = V(n\lambda_1) = \mathrm{Sym}_n(V(\lambda_1)),$$

  for some $n \geqslant 0$. Moreover, $\dim V_n = n + 1$ and $V_n^* = V_n$.
- The second exterior power of V_n decomposes into irreducibles, according to the Clebsch-Gordan rule:

$$V_n \wedge V_n = \bigoplus_{j \geqslant 0} V_{2n-2-4j}.$$

  These summands occur with multiplicity 1, and V_{2n-2} is always one of those summands.

PROPOSITION

Let K be an $\mathfrak{sl}_2(\mathbb{C})$-submodule of $V_n \wedge V_n$. TFAE:

1. *The variety $\mathcal{R}(V_n, K)$ consists only of $0 \in V_n^*$.*
2. *The $\mathbb{C}$-vector space $\mathcal{B}(V_n, K)$ is finite-dimensional.*
3. *The representation K contains V_{2n-2} as a direct summand.*

The $\operatorname{Sym}(V_n)$-modules $W(n) = \mathcal{B}(V_n, V_{2n-2})$ were studied by Weyman and Eisenbud (1990). We strengthen one of their results:

COROLLARY

For any $\mathfrak{sl}_2(\mathbb{C})$-submodule $K \subset V_n \wedge V_n$, the Koszul module $\mathcal{B}(V_n, K)$ is finite-dimensional over $\mathbb{C}$ if and only if $\mathcal{B}(V_n, K)$ is a quotient of $W(n)$.

Open problem: compute $\operatorname{Hilb}(W(n))$. The vanishing of $W_{n-2}(n)$, for all $n \geqslant 1$, would imply the generic Green Conjecture on free resolutions of canonical curves.

AUTOMORPHISM GROUPS OF FREE GROUPS

- Identify $(F_n)_{\text{ab}} = \mathbb{Z}^n$, and $\text{Aut}(\mathbb{Z}^n) = \text{GL}_n(\mathbb{Z})$. The morphism $\text{Aut}(F_n) \to \text{GL}_n(\mathbb{Z})$ is onto; thus, $\mathcal{A}(F_n) = \text{GL}_n(\mathbb{Z})$.
- Denote the Torelli group by $\text{IA}_n = \mathcal{I}_{F_n}$, and the Johnson–Andreadakis filtration by $J_n^s = F^s(\text{Aut}(F_n))$.
- Magnus [1934]: IA_n is generated by the automorphisms

$$\alpha_{ij} \colon \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_\ell \mapsto x_\ell \end{cases} \qquad \alpha_{ijk} \colon \begin{cases} x_i \mapsto x_i \cdot (x_j, x_k) \\ x_\ell \mapsto x_\ell \end{cases}$$

  with $1 \leqslant i \neq j \neq k \leqslant n$.
- Thus, $\text{IA}_1 = \{1\}$ and $\text{IA}_2 = \text{Inn}(F_2) \cong F_2$ are finitely presented.
- Krstić and McCool [1997]: IA_3 is not finitely presentable.
- It is not known whether IA_n admits a finite presentation for $n \geqslant 4$.

THE TORELLI GROUP OF F_n

Let $\mathcal{I}_{F_n} = J_n^1 = IA_n$ be the Torelli group of F_n. Recall we have an equivariant $\mathrm{GL}_n(\mathbb{Z})$-homomorphism,

$$J\colon \operatorname{gr}_F(\mathrm{IA}_n) \to \operatorname{Der}(\mathcal{L}_n),$$

In degree 1, this can be written as

$$J\colon \operatorname{gr}_F^1(\mathrm{IA}_n) \to H^* \otimes (H \wedge H),$$

where $H = (F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, viewed as a $\mathrm{GL}_n(\mathbb{Z})$-module via the defining representation. Composing with ι_F, we get a homomorphism

$$J \circ \iota_F\colon (\mathrm{IA}_n)_{\mathrm{ab}} \longrightarrow H^* \otimes (H \wedge H) .$$

THEOREM (ANDREADAKIS, COHEN–PAKIANATHAN, FARB, KAWAZUMI)

For each $n \geqslant 3$, the map $J \circ \iota_F$ is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism.

Thus, $H_1(\mathrm{IA}_n, \mathbb{Z})$ is free abelian, of rank $b_1(IA_n) = n^2(n-1)/2$.

We have a commuting diagram,

$$\begin{array}{ccccccccc}
 & & \mathrm{Inn}(F_n) & \xrightarrow{=} & \mathrm{Inn}(F_n) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{IA}_n & \longrightarrow & \mathrm{Aut}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow q & & \downarrow q & & \downarrow = & & \\
1 & \longrightarrow & \mathrm{OA}_n & \longrightarrow & \mathrm{Out}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

- Thus, $\mathrm{OA}_n = \widetilde{\mathcal{I}}_{F_n}$.
- Write the induced Johnson filtration on $\mathrm{Out}(F_n)$ as $\widetilde{J}^s_n = \pi(J^s_n)$.
- $\mathrm{GL}_n(\mathbb{Z})$ acts on $(\mathrm{OA}_n)_{\mathrm{ab}}$, and the outer Johnson homomorphism defines a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism

$$\widetilde{J} \circ \iota_{\widetilde{F}} \colon (\mathrm{OA}_n)_{\mathrm{ab}} \xrightarrow{\cong} H^* \otimes (H \wedge H) / H \,.$$

- Moreover, $\widetilde{J}^2_n = \mathrm{OA}'_n$, and we have an exact sequence

$$1 \longrightarrow F'_n \xrightarrow{\mathrm{Ad}} \mathrm{IA}'_n \longrightarrow \mathrm{OA}'_n \longrightarrow 1 \,.$$

DEEPER INTO THE JOHNSON FILTRATION

CONJECTURE (F. COHEN, A. HEAP, A. PETTET 2010)

If $n \geqslant 3$, $s \geqslant 2$, and $1 \leqslant i \leqslant n-2$, the cohomology group $H^i(J_n^s, \mathbb{Z})$ is not finitely generated.

We disprove this conjecture, at least rationally, in the case when $n \geqslant 5$, $s = 2$, and $i = 1$.

THEOREM

If $n \geqslant 5$, then $\dim_{\mathbb{Q}} H^1(J_n^2, \mathbb{Q}) < \infty$.

To start with, note that $J_n^2 = IA_n'$. Thus, it remains to prove that $b_1(\mathrm{IA}_n') < \infty$, i.e., $(IA_n'/IA_n'') \otimes \mathbb{Q}$ is finite-dimensional.

REPRESENTATIONS OF $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{h}$: the Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, with coordinates $t_1, \dots, t_n$.
- $\Delta = \{t_i - t_{i+1} \mid 1 \leqslant i \leqslant n-1\}$.
- $\lambda_i = t_1 + \cdots + t_i$.
- $V(\lambda)$: the irreducible, finite dimensional representation of $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\lambda = \sum_{i<n} a_i\lambda_i$, with $a_i \in \mathbb{Z}_{\geqslant 0}$.

Set $H_{\mathbb{C}} = H_1(F_n, \mathbb{C}) = \mathbb{C}^n$, and

$$V^* := H^1(\mathrm{OA}_n, \mathbb{C}) = H_{\mathbb{C}} \otimes (H_{\mathbb{C}}^* \wedge H_{\mathbb{C}}^*)/H_{\mathbb{C}}^*.$$

$$K^{\perp} := \ker\big(\cup\colon V^* \wedge V^* \to H^2(\mathrm{OA}_n, \mathbb{C})\big).$$

THEOREM (PETTET 2005)

Let $n \geqslant 4$. Set $\lambda = \lambda_2 + \lambda_{n-1}$ (so that $\lambda^* = \lambda_1 + \lambda_{n-2}$) and $\mu = \lambda_1 + \lambda_{n-2} + \lambda_{n-1}$. Then $V^* = V(\lambda^*)$ and $K^{\perp} = V(\mu)$, as $\mathfrak{sl}_n(\mathbb{C})$-modules.

THEOREM

For each $n \geqslant 4$, the resonance variety $\mathcal{R}(\mathrm{OA}_n)$ vanishes.

PROOF.

$2\lambda^* - \mu = t_1 - t_{n-1}$ is not a simple root. Thus, $\mathcal{R}(V, K) = 0$. □

REMARK

When $n = 3$, the proof breaks down, since $t_1 - t_2$ is a simple root. In fact, $K^\perp = V^* \wedge V^*$ in this case. Thus, $\mathcal{R}(V, K) = V^*$.

COROLLARY

For each $n \geqslant 4$, let $V = V(\lambda_2 + \lambda_{n-1})$ and let $K^\perp = V(\lambda_1 + \lambda_{n-2} + \lambda_{n-1}) \subset V^* \wedge V^*$ be the Pettet summand. Then $\dim \mathcal{B}(V, K) < \infty$ and $\dim \mathrm{gr}_q B(\mathrm{OA}_n) \leqslant \dim \mathcal{B}_q(V, K)$, for all $q \geqslant 0$.

Using now a result of Dimca–Papadima (2013) on the "geometric irreducibility" of representations of arithmetic groups, we obtain:

THEOREM

If $n \geqslant 4$, then $\mathcal{V}(\mathrm{OA}_n)$ is finite, and so $b_1(\mathrm{OA}'_n) < \infty$.

Finally,

THEOREM

If $n \geqslant 5$, then $b_1(\mathrm{IA}'_n) < \infty$.

PROOF.

The spectral sequence of the extension $1 \to F'_n \to \mathrm{IA}'_n \to \mathrm{OA}'_n \to 1$ gives rise to the exact sequence

$$H_1(F'_n, \mathbb{C})_{\mathrm{IA}'_n} \longrightarrow H_1(\mathrm{IA}'_n, \mathbb{C}) \longrightarrow H_1(\mathrm{OA}'_n, \mathbb{C}) \longrightarrow 0 .$$

The last term is finite-dimensional for all $n \geqslant 4$, by previous theorem. The first term is finite-dimensional for all $n \geqslant 5$, by nilpotency of the action of IA'_n on F'_n / F''_n. Conclusion follows. □

TORELLI GROUPS OF SURFACES

- Let Σ_g be a Riemann surface of genus g, and let $\mathcal{I}_g = \mathcal{I}_{\pi_1(\Sigma_g)}$.
- $\mathcal{I}_1$ is trivial, $\mathcal{I}_2$ is not finitely generated.
- So assume $g \geqslant 3$, in which case $\mathcal{I}_g$ is finitely generated.
- $\mathrm{Out}^+(\pi_1(\Sigma_g)) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is surjective; thus, there is a natural $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on $V = H_1(\mathcal{I}_g, \mathbb{C})$. This action extends to a rational irrep of $\mathrm{Sp}_{2g}(\mathbb{C})$, and thus, of $\mathfrak{sp}_{2g}(\mathbb{C})$.
- Dominant weights: $\lambda_i = t_1 + \cdots + t_i$, for $1 \leqslant i \leqslant g$.
- Let $V^* = H^1(\mathcal{I}_g, \mathbb{C})$, and let $K^\perp = \ker(\cup) \subset V^* \wedge V^*$.
- Hain (1997): $V^* = V(\lambda_3)$ and $K^\perp = V(2\lambda_2) \oplus V(0)$. Moreover, the decomposition of $V^* \wedge V^*$ into irreps is multiplicity-free.

THEOREM

$\mathcal{R}(\mathcal{I}_g) = 0$, for each $g \geqslant 4$.

PROOF.

- Simple roots: $\Delta = \{t_1 - t_2, t_2 - t_3, \dots, t_{g-1} - t_g, 2t_g\}$.
- The only $\beta \in \Delta$ for which $(\lambda_3, \beta) \neq 0$ is $\beta = t_3 - t_4$.
- Clearly, $2\lambda_3 - \beta = \lambda_2 + \lambda_4$ is not a dominant weight for $K^{\perp}$.
- Hence, $\mathcal{R}(V, K) = 0$.

□

Let $K_g \subset \mathcal{I}_g$ be the "Johnson kernel", i.e., the subgroup generated by Dehn twists about separating curves on Σ_g. The above result (and some more work) implies the following:

THEOREM (DIMCA–PAPADIMA 2013)

$H_1(K_g, \mathbb{C})$ is finite-dimensional, for each $g \geqslant 4$.