Resonance varieties

Alex Suciu

Northeastern University

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Scuola Superiore di Catania

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SUPPORT VARIETIES

- Let $\Bbbk$ be an algebraically closed field.
- Let S be a commutative, finitely generated $\Bbbk$-algebra.
- Let $\mathfrak{m}\mathrm{Spec}(S) = \mathrm{Hom}_{\Bbbk\text{-alg}}(S, \Bbbk)$ be the maximal spectrum of S.
- Let $E: \ \cdots \to E_i \xrightarrow{d_i} E_{i-1} \to \cdots \to E_0 \to 0$ be an S-chain complex.
- The *support varieties* of E are the subsets of $\mathfrak{m}\mathrm{Spec}(S)$ given by

$$\mathcal{W}^i_s(E) = \mathrm{supp}\Big(\bigwedge^s H_i(E) \Big).$$

- They depend only on the chain-homotopy equivalence class of E.
- For each $i \geqslant 0$, $\mathfrak{m}\mathrm{Spec}(S) = \mathcal{W}^i_0(E) \supseteq \mathcal{W}^i_1(E) \supseteq \mathcal{W}^i_2(E) \supseteq \cdots$.
- If all E_i are finitely generated S-modules, then the sets $\mathcal{W}^i_s(E)$ are Zariski closed subsets of $\mathfrak{m}\mathrm{Spec}(S)$.

HOMOLOGY JUMP LOCI

- The *homology jump loci* of the S-chain complex E are defined as

$$\mathcal{V}^i_s(E) = \{\mathfrak{m} \in \mathfrak{m}\mathrm{Spec}(S) \mid \dim_{\Bbbk} H_i(E \otimes_S S/\mathfrak{m}) \geqslant s\}.$$

- They depend only on the chain-homotopy equivalence class of E.
- For each $i \geqslant 0$, $\mathfrak{m}\mathrm{Spec}(S) = \mathcal{V}^i_0(E) \supseteq \mathcal{V}^i_1(E) \supseteq \mathcal{V}^i_2(E) \supseteq \cdots$.
- (Papadima–S. 2014) Suppose E is a chain complex of *free*, finitely generated S-modules. Then:
  - Each $\mathcal{V}^i_d(E)$ is a Zariski closed subset of $\mathfrak{m}\mathrm{Spec}(S)$.
  - For each q,

$$\bigcup_{i \leqslant q} \mathcal{V}^i_1(E) = \bigcup_{i \leqslant q} \mathcal{W}^i_1(E).$$

RESONANCE VARIETIES

- Let $A = \bigoplus_{i\geqslant 0} A^i$ be a commutative graded $\Bbbk$-algebra, with $A^0 = \Bbbk$.
- Let $a \in A^1$, and assume $a^2 = 0$ (this condition is redundant if $\operatorname{char}(\Bbbk) \neq 2$, by graded-commutativity of the multiplication in A).
- The *Aomoto complex* of A (with respect to $a \in A^1$) is the cochain complex of $\Bbbk$-vector spaces,

$$(A, \delta_a)\colon \ A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \xrightarrow{\ a\ } \cdots,$$

  with differentials given by $b \mapsto a \cdot b$, for $b \in A^i$.
- The *resonance varieties* of A are the sets

$$\mathcal{R}^i_s(A) = \{a \in A^1 \mid a^2 = 0 \text{ and } \dim_{\Bbbk} H^i(A, a) \geqslant s\}.$$

- If A is locally finite (i.e., $\dim_{\Bbbk} A^i < \infty$, for all $i \geqslant 1$), then the sets $\mathcal{R}^i_s(A)$ are Zariski closed cones inside the affine space A^1.

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^\vee$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.
- Define a cochain complex of free S-modules, $K(A) := (A^\bullet \otimes S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes s) = \sum_{j=1}^{n} e_j u \otimes s x_j$.
- The specialization of $(A \otimes S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).
- The cohomology support loci $R^i_s(A) = \mathrm{supp}(\bigwedge^s H^i(K(A)))$ are (closed) subvarieties of A^1.
- Both $\mathcal{R}^i_s(A)$ and $R^i_s(A)$ can be arbitrarily complicated (homogeneous) affine varieties.

EXAMPLE (EXTERIOR ALGEBRA)

Let $E = \bigwedge V$, where $V = \Bbbk^n$, and $S = \mathrm{Sym}(V)$. Then $K(E)$ is the Koszul complex on V. E.g., for $n = 3$:

$$S \xrightarrow{\begin{pmatrix} x_3 & -x_2 & x_1 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_2 & -x_1 & 0 \\ x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S \,.$$

This chain complex provides a free resolution $\varepsilon\colon K(E) \to \Bbbk$ of the trivial S-module $\Bbbk$. Hence,

$$\mathcal{R}^i_s(E) = \begin{cases} \{0\} & \text{if } s \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (NON-ZERO RESONANCE)

Let $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2\rangle$, and set $S = \Bbbk[x_1, x_2, x_3]$. Then

$$K(A):\quad S^2 \xrightarrow{\begin{pmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_s(A) = \begin{cases} \{x_3 = 0\} & \text{if } s = 1, \\ \{0\} & \text{if } s = 2 \text{ or } 3, \\ \emptyset & \text{if } s > 3. \end{cases}$$

EXAMPLE (NON-LINEAR RESONANCE)

Let $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4\rangle$. Then

$$K(A):\quad S^3 \xrightarrow{\begin{pmatrix} x_4 & 0 & 0 & -x_1 \\ 0 & x_3 & -x_2 & 0 \\ -x_2 & x_1 & x_4 & -x_3 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} S\,.$$

$$\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$$

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex.
- Fundamental group $\pi = \pi_1(X, x_0)$: a finitely generated, discrete group, with $\pi_{\text{ab}} \cong H_1(X, \mathbb{Z})$.
- Fix a field $\Bbbk$ with $\overline{\Bbbk} = \Bbbk$ (usually $\Bbbk = \mathbb{C}$), and let $S = \Bbbk[\pi_{\text{ab}}]$.
- Identify $\mathfrak{m}\text{Spec}(S)$ with the character group $\text{Char}(X) = \text{Hom}(\pi, \Bbbk^{\times})$, also denoted $\widehat{\pi} = \widehat{\pi_{\text{ab}}}$.
- The characteristic varieties of X are the homology jump loci of free S-chain complex $E = C_*(X^{\text{ab}}, \Bbbk)$:

$$\mathcal{V}^i_s(X, \Bbbk) = \{\rho \in \text{Char}(X) \mid \dim_{\Bbbk} H_i(X, \Bbbk_\rho) \geqslant s\}.$$

- Each set $\mathcal{V}^i_s(X, \Bbbk)$ is a subvariety of $\text{Char}(X)$.

EXAMPLE (CIRCLE)

Let $X = S^1$. We have $(S^1)^{\text{ab}} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_*((S^1)^{\text{ab}}) :\ 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For each $\rho \in \operatorname{Hom}(\mathbb{Z}, \Bbbk^\times) = \Bbbk^\times$, get a chain complex

$$C_*(\widetilde{S^1}) \otimes_{\mathbb{Z}\mathbb{Z}} \Bbbk_\rho :\ 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$. Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$

and $\mathcal{V}^i_s(S^1) = \emptyset$, otherwise.

EXAMPLE (TORUS)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\mathrm{Hom}(\mathbb{Z}^n, \Bbbk^\times) = (\Bbbk^\times)^n$. Then:

$$\mathcal{V}^i_s(T^n) = \begin{cases} \{1\} & \text{if } s \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

EXAMPLE (WEDGE OF CIRCLES)

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\mathrm{Hom}(F_n, \Bbbk^\times) = (\Bbbk^\times)^n$. Then:

$$\mathcal{V}^1_s(\bigvee^n S^1) = \begin{cases} (\Bbbk^\times)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

EXAMPLE (ORIENTABLE SURFACE OF GENUS $g > 1$)

$$\mathcal{V}^1_s(\Sigma_g) = \begin{cases} (\Bbbk^\times)^{2g} & \text{if } s < 2g-1, \\ \{1\} & \text{if } s = 2g-1, 2g, \\ \emptyset & \text{if } s > 2g. \end{cases}$$

- *Homotopy invariance:* If $X \simeq Y$, then $\mathcal{V}^i_s(Y, \Bbbk) \cong \mathcal{V}^i_s(X, \Bbbk)$.
- *Product formula:* $\mathcal{V}^i_1(X_1 \times X_2, \Bbbk) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1, \Bbbk) \times \mathcal{V}^q_1(X_2, \Bbbk)$.
- *Degree 1 interpretation:* The sets $\mathcal{V}^1_s(X, \Bbbk)$ depend only on $\pi = \pi_1(X)$—in fact, only on π/π''. Write them as $\mathcal{V}^1_s(\pi, \Bbbk)$.
- *Functoriality:* If $\varphi\colon \pi \twoheadrightarrow G$ is an epimorphism, then $\hat{\varphi}\colon \widehat{G} \hookrightarrow \widehat{\pi}$ restricts to an embedding $\mathcal{V}^1_s(G, \Bbbk) \hookrightarrow \mathcal{V}^1_s(\pi, \Bbbk)$, for each s.
- *Universality:* Given any subvariety $W \subset (\Bbbk^{\times})^n$, there is a finitely presented group π such that $\pi_{ab} = \mathbb{Z}^n$ and $\mathcal{V}^1_1(\pi, \Bbbk) = W$.
- *Alexander invariant interpretation:* Let $X^{ab} \to X$ be the maximal abelian cover. View $H_*(X^{ab}, \Bbbk)$ as a module over $S = \Bbbk[\pi_{ab}]$. Then:

$$\bigcup_{j\leqslant i} \mathcal{V}^j_1(X, \Bbbk) = \operatorname{supp}\Big(\bigoplus_{j\leqslant i} H_j(X^{ab}, \Bbbk)\Big).$$

THE TANGENT CONE THEOREM

- The *resonance varieties* of X (with coefficients in $\Bbbk$) are the loci $\mathcal{R}^i_d(X,\Bbbk)$ associated to the cohomology algebra $A = H^*(X,\Bbbk)$.
- Each set $\mathcal{R}^i_s(X) := \mathcal{R}^i_s(X,\mathbb{C})$ is a homogeneous subvariety of $H^1(X,\mathbb{C}) \cong \mathbb{C}^n$, where $n = b_1(X)$.
- Recall that $\mathcal{V}^i_s(X) := \mathcal{V}^i_s(X,\mathbb{C})$ is a subvariety of $H^1(X,\mathbb{C}^\times) \cong (\mathbb{C}^\times)^n \times \operatorname{Tors}(H_1(X,\mathbb{Z}))$.
- (Libgober 2002) $\operatorname{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X)$.
- Given a subvariety $W \subset H^1(X,\mathbb{C}^\times)$, let $\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$.
- (Dimca–Papadima–S. 2009) $\tau_1(W)$ is a finite union of rationally defined linear subspaces, and $\tau_1(W) \subseteq \operatorname{TC}_1(W)$.
- Thus, $\tau_1(\mathcal{V}^i_s(X)) \subseteq \operatorname{TC}_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X)$.

FORMALITY

- X is *formal* if there is a zig-zag of cdga quasi-isomorphisms from $(A_{\mathrm{PL}}(X,\mathbb{Q}),d)$ to $(H^*(X,\mathbb{Q}),0)$.
- X is k*-formal* (for some $k \geqslant 1$) if each of these morphisms induces an iso in degrees up to k, and a monomorphism in degree $k+1$.
- X is 1-formal if and only if $\pi = \pi_1(X)$ is 1-formal, i.e., its Malcev Lie algebra, $\mathfrak{m}(\pi) = \mathrm{Prim}(\widehat{\mathbb{Q}\pi})$, is quadratic.
- For instance, compact Kähler manifolds and complements of hyperplane arrangements are formal.
- (Dimca–Papadima–S. 2009) Let X be a 1-formal space. Then, for each $s > 0$,

$$\tau_1(\mathcal{V}^1_s(X)) = \mathrm{TC}_1(\mathcal{V}^1_s(X)) = \mathcal{R}^1_s(X).$$

  Consequently, $\mathcal{R}^1_s(X)$ is a finite union of rationally defined linear subspaces in $H^1(X,\mathbb{C})$.

This theorem yields a very efficient formality test.

EXAMPLE

Let $\pi = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_4][x_2^{-2}, x_3], [x_1^{-1}, x_3][x_2, x_4] \rangle$. Then $\mathcal{R}_1^1(\pi) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$ splits into linear subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, π is *not* 1-formal.

EXAMPLE

Let $F(\Sigma_g, n)$be the configuration space of n labeled points of a Riemann surface of genus g (a smooth, quasi-projective variety).

Then $\pi_1(F(\Sigma_g, n)) = P_{g,n}$, the pure braid group on n strings on Σ_g. Compute:

$$\mathcal{R}_1^1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \leqslant i < j < n \end{array} \right\}$$

For $n \geqslant 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

APPLICATIONS OF COHOMOLOGY JUMP LOCI

- Obstructions to formality and (quasi-) projectivity
  - Right-angled Artin groups and Bestvina–Brady groups
  - 3-manifold groups, Kähler groups, and quasi-projective groups
- Homology of finite, regular abelian covers
  - Homology of the Milnor fiber of an arrangement
  - Rational homology of smooth, real toric varieties
- Homological and geometric finiteness of regular abelian covers
  - Bieri–Neumann–Strebel–Renz invariants
  - Dwyer–Fried invariants
- Resonance varieties and representations of Lie algebras
  - Homological finiteness in the Johnson filtration of automorphism groups
- Lower central series and Chen Lie algebras
  - The resonance–Chen ranks formula

QUASI-PROJECTIVE VARIETIES

THEOREM (ARAPURA 1997, ..., BUDUR–WANG 2015)

Let X be a smooth, quasi-projective variety. Then each $\mathcal{V}^i_s(X)$ is a finite union of torsion-translated subtori of $\mathrm{Char}(X)$.

THEOREM (DIMCA–PAPADIMA–S. 2009)

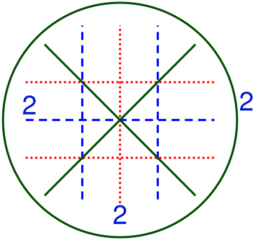
Let X be a smooth, quasi-projective variety. If X is 1-formal, then the (non-zero) irreducible components of $\mathcal{R}^1_1(X)$ are linear subspaces of $H^1(X, \mathbb{C})$ which intersect pairwise only at 0. Each such component L_α is p-isotropic (i.e., the restriction of $\cup_X$ to L_α has rank p), with $\dim L_\alpha \geqslant 2p+2$, for some $p = p(\alpha) \in \{0, 1\}$, and

$$\mathcal{R}^1_s(X) = \{0\} \cup \bigcup_{\alpha : \dim L_\alpha > s + p(\alpha)} L_\alpha .$$

- *If X is compact, then X is 1-formal, and each L_α is 1-isotropic.*
- *If $W_1(H^1(X, \mathbb{C})) = 0$, then X is 1-formal, and each L_α is 0-isotropic.*

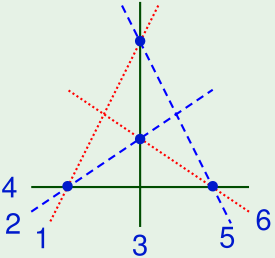
HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement in $\mathbb{C}^3$, and identify $H^1(M(\mathcal{A}), \Bbbk) = \Bbbk^n$, with basis dual to the meridians.
- The resonance varieties $\mathcal{R}^1_s(\mathcal{A}, \Bbbk) := \mathcal{R}^1_s(M(\mathcal{A}), \Bbbk) \subset \Bbbk^n$ lie in the hyperplane $\{x \in \Bbbk^n \mid x_1 + \cdots + x_n = 0\}$.
- $\mathcal{R}^1(\mathcal{A}) = \mathcal{R}^1_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $\mathbb{C}^n$, described in work of Falk, Cohen–Suciu, Libgober–Yuzvinsky, Falk–Yuzvinsky.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}^1_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$.



- Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geqslant 3$ gives rise to a *local* component of $\mathcal{R}^1(\mathcal{A})$, of dimension $k-1$.
- More generally, every k-*multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k-1$, and all components of $\mathcal{R}^1(\mathcal{A})$ arise in this way.
- The resonance varieties $\mathcal{R}^1(\mathcal{A}, \Bbbk)$ can be more complicated, e.g., they may have non-linear components.

EXAMPLE (BRAID ARRANGEMENT $\mathcal{A}_4$)



$\mathcal{R}^1(\mathcal{A}) \subset \mathbb{C}^6$ has 4 components coming from the triple points, and one component from the above 3-net:

$$L_{124} = \{x_1 + x_2 + x_4 = x_3 = x_5 = x_6 = 0\},$$
$$L_{135} = \{x_1 + x_3 + x_5 = x_2 = x_4 = x_6 = 0\},$$
$$L_{236} = \{x_2 + x_3 + x_6 = x_1 = x_4 = x_5 = 0\},$$
$$L_{456} = \{x_4 + x_5 + x_6 = x_1 = x_2 = x_3 = 0\},$$
$$L = \{x_1 + x_2 + x_3 = x_1 - x_6 = x_2 - x_5 = x_3 - x_4 = 0\}.$$

- Let $\text{Hom}(\pi_1(M), \mathbb{k}^\times) = (\mathbb{k}^\times)^n$ be the character torus.
- The characteristic variety $\mathcal{V}^1(\mathcal{A}, \mathbb{k}) := \mathcal{V}^1_1(M(\mathcal{A}), \mathbb{k}) \subset (\mathbb{k}^\times)^n$ lies in the substorus $\{t \in (\mathbb{k}^\times)^n \mid t_1 \cdots t_n = 1\}$.
- $\mathcal{V}^1(\mathcal{A}) = \mathcal{V}^1(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^\times)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}^1(\mathcal{A})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}^1(\mathcal{A})$.
- All components of $\mathcal{V}^1(\mathcal{A})$ passing through the origin $1 \in (\mathbb{C}^\times)^n$ arise in this way (and thus, are combinatorially determined).
- In general, though, there are translated subtori in $\mathcal{V}^1(\mathcal{A})$.

QUESTION

Is $\mathcal{V}^1(\mathcal{A})$ combinatorially determined?

TORIC COMPLEXES AND RAAGS

- Let L be a simplicial complex on n vertices.
- The *toric complex* T_L is the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L. That is:
  - $S^1 = e^0 \cup e^1$.
  - $T^n = (S^1)^{\times n}$, with product cell structure:

$$(k-1)\text{-simplex } \sigma = \{i_1, \dots, i_k\} \quad \rightsquigarrow \quad k\text{-cell } e^{\sigma} = e^1_{i_1} \times \cdots \times e^1_{i_k}$$

  - $T_L = \bigcup_{\sigma \in L} e^{\sigma}$.
- Examples:
  - $T_{\emptyset} = *$
  - $T_{n \text{ points}} = \bigvee^n S^1$
  - $T_{\partial\Delta^{n-1}} = (n-1)$-skeleton of T^n
  - $T_{\Delta^{n-1}} = T^n$

- $\pi_1(T_L)$ is the *right-angled Artin group* associated to the graph $\Gamma = L^{(1)}$:

$$G_L = G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle.$$

- If $\Gamma = \overline{K}_n$ then $G_\Gamma = F_n$, while if $\Gamma = K_n$, then $G_\Gamma = \mathbb{Z}^n$.
- If $\Gamma = \Gamma' \coprod \Gamma''$, then $G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$.
- If $\Gamma = \Gamma' * \Gamma''$, then $G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$.
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ. (Davis–Charney 1995, Meier–VanWyk 1995)
- $H^*(T_L, \mathbb{Z})$ is the *exterior Stanley-Reisner ring* of L, with generators the duals v^*, and relations the monomials corresponding to the missing simplices of L.
- If $H^*(T_K, \mathbb{Z}) \cong H^*(T_L, \mathbb{Z})$, then $K \cong L$. (Stretch 2017)
- T_L is formal, and so G_L is 1-formal. (Notbohm–Ray 2005)

Identify $H^1(T_L,\mathbb{C})=\mathbb{C}^{\mathsf{V}}$, the $\mathbb{C}$-vector space with basis $\{v \mid v\in \mathsf{V}\}$.

THEOREM (PAPADIMA–S. 2010)

$$\mathcal{R}^i_s(T_L,\Bbbk)=\bigcup_{\substack{\mathsf{W}\subset\mathsf{V}\\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}}\dim_{\Bbbk}\widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\Bbbk)\geqslant s}}\mathbb{C}^{\mathsf{W}},$$

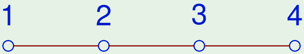
where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K\subseteq L$.

In particular (PS06):

$$\mathcal{R}^1_1(G_\Gamma,\Bbbk)=\bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V}\\ \Gamma_{\mathsf{W}}\text{ disconnected}}}\Bbbk^{\mathsf{W}}.$$

Similar formula holds for $\mathcal{V}^i_s(T_L,\Bbbk)$, with $\Bbbk^{\mathsf{W}}$ replaced by $(\Bbbk^\times)^{\mathsf{W}}$.

EXAMPLE

$\Gamma =$ 

Maximal disconnected subgraphs: $\Gamma_{\{134\}}$ and $\Gamma_{\{124\}}$. Thus:

$$\mathcal{R}_1(G_\Gamma) = \mathbb{C}^{\{134\}} \cup \mathbb{C}^{\{124\}}.$$

Note that: $\mathbb{C}^{\{134\}} \cap \mathbb{C}^{\{124\}} = \mathbb{C}^{\{14\}} \neq \{0\}$ Since G_Γ is 1-formal, G_Γ is *not* a quasi-projective group.

THEOREM (DPS09)

The following are equivalent:

1. G_Γ *is a quasi-projective group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. G_Γ *is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$