The algebra and topology of partial products of circles

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Partial product construction

Input:

- K, a simplicial complex on $[n] = \{1, \dots, n\}$.
- (X, A), a pair of topological spaces, $A \neq \emptyset$.

Output:

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma \subset X^{\times n}$$

where $(X, A)^\sigma = \{x \in X^{\times n} \mid x_i \in A \text{ if } i \notin \sigma\}$.
Interpolates between

- $\mathcal{Z}_\emptyset(X, A) = \mathcal{Z}_K(A, A) = A^{\times n}$ and
- $\mathcal{Z}_{\Delta^{n-1}}(X, A) = \mathcal{Z}_K(X, X) = X^{\times n}$

Examples:

- $\mathcal{Z}_{n \text{ points}}(X, *) = \bigvee^n X$ (wedge)
- $\mathcal{Z}_{\partial\Delta^{n-1}}(X, *) = T^n X$ (fat wedge)

Properties:

- $L \subset K$ subcomplex $\Rightarrow \mathcal{Z}_L(X,A) \subset \mathcal{Z}_K(X,A)$ subspace.
- (X,A) pair of (finite) CW-complexes $\Rightarrow \mathcal{Z}_K(X,A)$ is a (finite) CW-complex.
- $\mathcal{Z}_{K*L}(X,A) \cong \mathcal{Z}_K(X,A) \times \mathcal{Z}_L(X,A)$.
- $f\colon (X,A) \to (Y,B)$ continuous map $\Rightarrow f^{\times n}\colon X^{\times n} \to Y^{\times n}$ restricts to a continuous map $\mathcal{Z}^f\colon \mathcal{Z}_K(X,A) \to \mathcal{Z}_K(Y,B)$.
- Consequently, $(X,A) \simeq (Y,B) \Rightarrow \mathcal{Z}_K(X,A) \simeq \mathcal{Z}_K(Y,B)$.
- (Strickland) $f\colon K \to L$ simplicial $\rightsquigarrow \mathcal{Z}_f\colon \mathcal{Z}_K(X,A) \to \mathcal{Z}_L(X,A)$ continuous (if X connected topological monoid, A submonoid).
- (Denham–S. 2005) If $(M, \partial M)$ is a compact manifold of dim d, and K is a PL-triangulation of S^m on n vertices, then $\mathcal{Z}_K(M, \partial M)$ is a compact manifold of dim $(d-1)n + m + 1$.
- (Bosio–Meersseman 2006) If K is a polytopal triangulation of S^m, then $\mathcal{Z}_K(D^2, S^1)$ if $n+m+1$ is even, or $\mathcal{Z}_K(D^2, S^1) \times S^1$ if $n+m+1$ is odd, is a complex manifold.

Toric complexes and right-angled Artin groups

Definition

Let L be simplicial complex on n vertices. The associated *toric complex*, T_L, is the subcomplex of the n-torus obtained by deleting the cells corresponding to the missing simplices of L, i.e.,

$$T_L = \mathcal{Z}_L(S^1, *).$$

- k-cells in $T_L \longleftrightarrow (k-1)$-simplices in L.
- $C_*^{\text{CW}}(T_L)$ is a subcomplex of $C_*^{\text{CW}}(T^n)$; thus, all $\partial_k = 0$, and

$$H_k(T_L, \mathbb{Z}) = C_{k-1}^{\text{simplicial}}(L, \mathbb{Z}) = \mathbb{Z}^{\#\,(k-1)\text{-simplices of } L}.$$

- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* $\bigwedge V^*/J_L$, where
  - V is the free $\Bbbk$-module on the vertex set of L
  - $\bigwedge V^*$ is the exterior algebra on dual of V,
  - J_L is the ideal generated by all monomials, $t_\sigma = v_{i_1}^* \cdots v_{i_k}^*$ corresponding to simplices $\sigma = \{v_{i_1}, \dots, v_{i_k}\}$ not belonging to L.

Right-angled Artin groups

Definition

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a (finite, simple) graph. The corresponding *right-angled Artin group* is

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle.$$

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$; $\quad \Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$; $\quad \Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$
- $\Gamma \cong \Gamma' \Leftrightarrow G_\Gamma \cong G_{\Gamma'}$
  (Kim–Makar-Limanov–Neggers–Roush 1980)
- $\pi_1(T_L) = G_\Gamma$, where $\Gamma = L^{(1)}$.
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
  (Davis–Charney 1995, Meier–VanWyk 1995)
- $A := H^*(G_\Gamma, \Bbbk) = \bigwedge V^*/J_\Gamma$, where J_Γ is quadratic monomial ideal
  $\Rightarrow$ A is a Koszul algebra (Fröberg 1975).

Formality

Definition (Sullivan)

A space X is *formal* if its minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$.

Definition (Quillen)

A group G is 1*-formal* if its Malcev Lie algebra, $\mathfrak{m}_G = \text{Prim}(\widehat{\mathbb{Q}G})$, is a (complete, filtered) quadratic Lie algebra.

Theorem (Sullivan)

If X formal, then $\pi_1(X)$ is 1-formal.

Theorem (Notbohm–Ray 2005)

T_L is formal, and so G_Γ is 1-formal.

Associated graded Lie algebra

Let G be a finitely-generated group. Define:

- *LCS series*: $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k \triangleright \cdots$, where $G_{k+1} = [G_k, G]$
- *LCS quotients*: $\operatorname{gr}_k G = G_k / G_{k+1}$ (f.g. abelian groups)
- *LCS ranks*: $\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)$
- *Associated graded Lie algebra*: $\operatorname{gr}(G) = \bigoplus_{k\geq 1} \operatorname{gr}_k(G)$, with Lie bracket $[\,,\,]\colon L_i \times L_j \to L_{i+j}$ induced by group commutator.

Example (Witt, Magnus)

Let $G = F_n$ (free group of rank n).
Then $\operatorname{gr} G = \operatorname{Lie}_n$ (free Lie algebra of rank n), with LCS ranks given by

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = 1 - nt.$$

Explicitly: $\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}$, where μ is Möbius function.

Holonomy Lie algebra

Definition (Chen)

The *holonomy Lie algebra* of G is the quadratic, graded Lie algebra

$$\mathfrak{h}_G = \mathrm{Lie}(H_1)/\mathrm{ideal}(\mathrm{im}(\nabla))$$

where $H_i = H_1(G, \mathbb{Z})$, and $\nabla \colon H_2 \to H_1 \wedge H_1 = \mathrm{Lie}_2(H_1)$ is the comultiplication map.

Properties:

- $U(\mathfrak{h} \otimes \mathbb{Q}) \cong \mathrm{Ext}_A(\mathbb{Q}, \mathbb{Q})$, for $G = \pi_1(X)$ and $A = H^*(X, \mathbb{Q})$.
- There is a canonical epimorphism $\mathfrak{h}_G \twoheadrightarrow \mathrm{gr}(G)$.
- If G is 1-formal, then $\mathfrak{h}_G \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$.

Example

$G = F_n$, then clearly $\mathfrak{h}_G = \mathrm{Lie}_n$, and so $\mathfrak{h}_G = \mathrm{gr}(G)$.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ graph, and $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma) t^k$ its clique polynomial.

Theorem (Duchamp–Krob 1992, Papadima–S. 2006)

For $G = G_\Gamma$:

1. $\operatorname{gr}(G) \cong \mathfrak{h}_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t).$$

Idea of proof:

1. $A = \bigwedge V^*/J_\Gamma \Rightarrow \mathfrak{h}_G = L_\Gamma := \operatorname{Lie}(\mathsf{V})/([v, w] = 0 \text{ if } \{v, w\} \in \mathsf{E})$.
2. Shelton–Yuzvinsky: $U(L_\Gamma) = A^!$ (Koszul dual).
3. Koszul duality: $\operatorname{Hilb}(A^!, t) \cdot \operatorname{Hilb}(A, -t) = 1$.
4. Computation independent of coefficient field $\Rightarrow \mathfrak{h}_G$ torsion-free.
5. But $\mathfrak{h}_G \twoheadrightarrow \operatorname{gr}(G)$ is iso over $\mathbb{Q}$ (by 1-formality) $\Rightarrow$ iso over $\mathbb{Z}$.
6. LCS formula follows from (3) and PBW.

Chen Lie algebras

Definition

The *Chen Lie algebra* of a (finitely generated) group G is $\operatorname{gr}(G/G'')$, i.e., the assoc. graded Lie algebra of its maximal metabelian quotient. Write $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ for the Chen ranks.

Facts:

- $\operatorname{gr}(G) \twoheadrightarrow \operatorname{gr}(G/G'')$, and so $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$.
- The map $\mathfrak{h}_G \twoheadrightarrow \operatorname{gr}(G)$ induces epimorphism $\mathfrak{h}_G/\mathfrak{h}''_G \twoheadrightarrow \operatorname{gr}(G/G'')$.
- (P.–S. 2004) If G is 1-formal, then $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G/G'') \otimes \mathbb{Q}$.

Example (Chen)

$$\theta_k(F_n) = \binom{n+k-2}{k}(k-1), \quad \text{for all } k \geq 2.$$

The Chen Lie algebra of a RAAG

Theorem (Papadima–S. 2006)

For $G = G_\Gamma$:

1. $\operatorname{gr}(G/G'') \cong \mathfrak{h}_G/\mathfrak{h}''_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W}\subset \mathsf{V}:\ |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

Idea of proof:

1. Write $A := H^*(G, \Bbbk) = E/J_\Gamma$, where $E = \bigwedge_{\Bbbk}(v_1^*, \dots, v_n^*)$.
2. Write $\mathfrak{h} = \mathfrak{h}_G \otimes \Bbbk$.
3. By Fröberg and Löfwall (2002)
$$\left(\mathfrak{h}'/\mathfrak{h}''\right)_k \cong \operatorname{Tor}^E_{k-1}(A, \Bbbk)_k, \quad \text{for } k \geq 2$$
4. By Aramova–Herzog–Hibi & Aramova–Avramov–Herzog (97-99):
$$\sum_{k\geq 2} \dim_{\Bbbk} \operatorname{Tor}^E_{k-1}(E/J_\Gamma, \Bbbk)_k = \sum_{i\geq 1} \dim_{\Bbbk} \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} \cdot \left(\frac{t}{1-t}\right)^{i+1},$$
where $S = \Bbbk[x_1, \dots, x_n]$ and $I_\Gamma =$ ideal $\langle x_i x_j \mid \{v_i, v_j\} \notin \mathsf{E}\rangle$.
5. By Hochster (1977):
$$\dim_{\Bbbk} \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} = \sum_{\mathsf{W}\subset \mathsf{V}\colon\, |\mathsf{W}|=i+1} \dim_{\Bbbk} \widetilde{H}_0(\Gamma_{\mathsf{W}}, \Bbbk) = c_{i+1}(\Gamma).$$
6. The answer is independent of $\Bbbk \Rightarrow \mathfrak{h}_G/\mathfrak{h}''_G$ is torsion-free.
7. Using formality of G_Γ, together with $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G/G'') \otimes \mathbb{Q}$ ends the proof.

Example

Let Γ be a pentagon, and Γ' a square with an edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 - 5t + 5t^2$, and so

$$\phi_k(G_\Gamma) = \phi_k(G_{\Gamma'}), \quad \text{for all } k \geq 1.$$

- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so

$$\theta_k(G_\Gamma) \neq \theta_k(G_{\Gamma'}), \quad \text{for } k \geq 4.$$

Artin kernels

Definition

Given a graph Γ, and an epimorphism $\chi\colon G_\Gamma \twoheadrightarrow \mathbb{Z}$, the corresponding *Artin kernel* is the group

$$N_\chi = \ker(\chi\colon G_L \to \mathbb{Z})$$

Note that $N_\chi = \pi_1(T_L^\chi)$, where $T_L^\chi \to T_L$ is the regular $\mathbb{Z}$-cover defined by χ. A classifying space for N_χ is $T_{\Delta_\Gamma}^\chi$, where $\Gamma = L^{(1)}$.

Noteworthy is the case when χ is the “diagonal" homomorphism $\nu\colon G_L \twoheadrightarrow \mathbb{Z}$, which assigns to each vertex the weight 1.
The corresponding Artin kernel, $N_\Gamma = N_\nu$, is called the *Bestvina–Brady group* associated to Γ.

Stallings, Bieri, Bestvina and Brady: geometric and homological finiteness properties of $N_\Gamma \longleftrightarrow$ topology of Δ_Γ, e.g.:

- N_Γ is finitely generated $\iff$ Γ is connected
- N_Γ is finitely presented $\iff$ Δ_Γ is simply-connected.

More generally, it follows from Meier–Meinert–VanWyk (1998) and Bux–Gonzalez (1999) that:

Theorem

Assume L is a flag complex. Let $\mathsf{W} = \{v \in \mathsf{V} \mid \chi(v) \neq 0\}$ *be the support of* χ*. Then:*

1. N_χ *is finitely generated* $\iff$ L_{W} *is connected, and,* $\forall\, v \in \mathsf{V} \setminus \mathsf{W}$*, there is a* $w \in \mathsf{W}$ *such that* $\{v, w\} \in L$*.*
2. N_χ *is finitely presented* $\iff$ L_{W} *is* 1*-connected and,* $\forall\, \sigma \in L_{\mathsf{V}\setminus\mathsf{W}}$*, the space* $\operatorname{lk}_{L_{\mathsf{W}}}(\sigma) = \{\tau \in L_{\mathsf{W}} \mid \tau \cup \sigma \in L\}$ *is* $(1 - |\sigma|)$*-acyclic.*

Theorem (P.–S. 2009)

Let Γ be a graph, and N_χ and Artin kernel.

1. *If $H_1(N_\chi, \mathbb{Q})$ is a trivial $\mathbb{Q}\mathbb{Z}$-module, then N_χ is finitely generated.*
2. *If both $H_1(N_\chi, \mathbb{Q})$ and $H_2(N_\chi, \mathbb{Q})$ have trivial $\mathbb{Z}$-action, then N_χ is 1-formal.*

Thus, if Γ is connected, and $H_1(\Delta_\Gamma, \mathbb{Q}) = 0$, then N_Γ is 1-formal.

Theorem (P.–S. 2009)

Suppose $H_1(N, \mathbb{Q})$ has trivial $\mathbb{Z}$-action. Then, both $\operatorname{gr}(N)$ and $\operatorname{gr}(N/N'')$ are torsion-free, with graded ranks, ϕ_k and θ_k, given by

$$\prod_{k=1}^{\infty}(1 - t^k)^{\phi_k} = \frac{P_\Gamma(-t)}{1 - t},$$

$$\sum_{k=2}^{\infty}\theta_k t^k = Q_\Gamma\left(\frac{t}{1 - t}\right).$$

Resonance varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$).
Let $\Bbbk$ be a field; if char $\Bbbk = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion.
Let $A = H^*(X, \Bbbk)$. Then: $a \in A^1 \Rightarrow a^2 = 0$. Thus, get cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

Definition (Falk 1997, Matei–S. 2000)

The *resonance varieties* of X (over $\Bbbk$) are the algebraic sets

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

- $\mathcal{R}^i_d$ are homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$
- $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots \supseteq \mathcal{R}^i_{b_i+1} = \emptyset$, where $b_i = b_i(X, \Bbbk)$.
- $\mathcal{R}^1_d(X, \Bbbk)$ depends only on $G = \pi_1(X)$, so denote it by $\mathcal{R}_d(G, \Bbbk)$.

Resonance of toric complexes

Recall $A = H^*(T_L, \Bbbk)$ is the exterior Stanley-Reisner ring of L. Using a formula of Aramova, Avramov, and Herzog (1999), we prove:

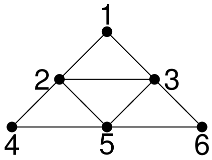
Theorem (Papadima–S. 2009)

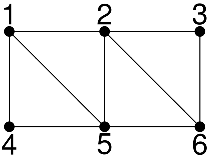
$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W}\subset \mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus \mathsf{W}}} \dim_{\Bbbk} \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma),\Bbbk)\geq d}} \Bbbk^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{R}^1_1(G_\Gamma, \Bbbk) = \bigcup_{\substack{\mathsf{W}\subseteq \mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \Bbbk^{\mathsf{W}}.$$





Example

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, G_Γ and $G_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}_1(G_\Gamma, \Bbbk) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}_1(G_{\Gamma'}, \Bbbk) = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

Kähler manifolds

Definition

A compact, connected, complex manifold M is called a *Kähler manifold* if M admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form.

Examples: Riemann surfaces, $\mathbb{CP}^n$, and, more generally, smooth, complex projective varieties.

Definition

A group G is a *Kähler group* if $G = \pi_1(M)$, for some compact Kähler manifold M.

G is *projective* if M is actually a smooth projective variety.

- G finite $\Rightarrow$ G is a projective group (Serre 1958).
- G_1, G_2 Kähler groups $\Rightarrow$ $G_1 \times G_2$ is a Kähler group
- G Kähler group, $H < G$ finite-index subgroup $\Rightarrow$ H is a Kähler gp

Problem (Serre 1958)

Which finitely presented groups are Kähler (or projective) groups?

The Kähler condition puts strong restrictions on M:

1. $H^*(M,\mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of M are even
3. M is formal, i.e., $(\Omega(M), d) \simeq (H^*(M,\mathbb{R}), 0)$ (Deligne–Griffiths–Morgan–Sullivan 1975)

The Kähler condition also puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. G is 1-formal, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is quadratic
3. G cannot split non-trivially as a free product (Gromov 1989)

Quasi-Kähler manifolds

Definition

A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact Kähler manifold and D is a divisor with normal crossings.

Similar definition for X quasi-projective.
The notions of quasi-Kähler group and quasi-projective group are defined as above.

- X quasi-projective $\Rightarrow H^*(X, \mathbb{Z})$ has a mixed Hodge structure (Deligne 1972–74)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal (Brieskorn 1973)
- X quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal (Kohno 1983)

Resonance varieties of quasi-Kähler manifolds

Theorem (D.–P.–S. 2009)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}_1(G)$. If G is 1-formal, then

1. *Each L_α is a p-isotropic linear subspace of $H^1(G, \mathbb{C})$, with* $\dim L_\alpha \geq 2p + 2$, *for some $p = p(\alpha) \in \{0, 1\}$.*
2. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
3. $\mathcal{R}_d(G) = \{0\} \cup \bigcup_\alpha L_\alpha$, *where the union is over all α for which* $\dim L_\alpha > d + p(\alpha)$.

Furthermore,

4. *If X is compact Kähler, then G is 1-formal, and each L_α is 1-isotropic.*
5. *If X is a smooth, quasi-projective variety, and $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Here we used the following

Definition

A non-zero subspace $U \subseteq H^1(G,\mathbb{C})$ is *p-isotropic* with respect to

$$\cup_G \colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$$

if the restriction of $\cup_G$ to $U \wedge U$ has rank p.

Example

Let C be a smooth complex curve with $\chi(C) < 0$. Then

$$\mathcal{R}^1_1(\pi_1(C),\mathbb{C}) = H^1(C,\mathbb{C})$$

and this space is either 1- or 0-isotropic, according to whether C is compact or not.

Theorem (Dimca–Papadima–S. 2009)

The following are equivalent:

1. G_Γ *is a quasi-Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. G_Γ *is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Example

Let Γ be a linear path on 4 vertices. The maximal disconnected subgraphs are $\Gamma_{\{134\}}$ and $\Gamma_{\{124\}}$. Thus:

$$\mathcal{R}_1(G_\Gamma, \mathbb{C}) = \mathbb{C}^{\{134\}} \cup \mathbb{C}^{\{234\}}.$$

But $\mathbb{C}^{\{134\}} \cap \mathbb{C}^{\{234\}} = \mathbb{C}^{\{14\}}$, which is a non-zero subspace. Thus, G_Γ is not a quasi-Kähler group.

Theorem (D.–P.–S. 2008)

For a Bestvina–Brady group $N_\Gamma = \ker(\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, *the following are equivalent:*

1. N_Γ *is a quasi-Kähler group*
2. Γ *is either a tree, or* $\Gamma = K_{n_1,\dots,n_r}$, *with some* $n_i = 1$, *or all* $n_i \geq 2$ *and* $r \geq 3$.

1. N_Γ *is a Kähler group*
2. $\Gamma = K_{2r+1}$
3. $N_\Gamma = \mathbb{Z}^{2r}$

Example

$\Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2$

$N_\Gamma =$ the Stallings group $= \pi_1(\mathbb{CP}^2 \setminus \{6 \text{ lines}\})$

N_Γ is finitely presented, but $H_3(N_\Gamma, \mathbb{Z})$ has infinite rank, so N_Γ not FP_3.

Hyperplane arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$, and group $G = G(\mathcal{A}) = \pi_1(X)$.

1. X is a smooth, quasi-projective variety, and so G is a quasi-projective group.
2. X is formal, and so $G = \pi_1(X)$ is 1-formal.
3. $A = H^*(X,\mathbb{Z})$ is the Orlik-Solomon algebra, determined by the intersection lattice, $L(\mathcal{A})$.
4. The resonance variety $\mathcal{R}^1_1(X,\mathbb{C})$ depends only on a generic section $\mathcal{A}' = \{\ell_1, \dots \ell_n\}$ in $\mathbb{C}^2$.
    - Each component is a linear subspace.
    - There are “local" components, corresponding to points where $k \geq 3$ lines in $\mathcal{A}'$ meet (these have dim $= k-1$).
    - There are also non-local components, corresponding to certain “multinets" (these have dim $= 2$ or 3).

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$.

Theorem (S. 2009)

The following are equivalent:

1. *G is a Kähler group.*
2. *G is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an even number of lines in general position.*

Theorem (S. 2009)

The following are equivalent:

1. *G is a right-angled Artin group.*
2. *G is a finite direct product of finitely generated free groups.*
3. *The multiplicity graph $\Gamma(\mathcal{A})$ is a forest.*

Σ-invariants

G finitely generated group $\rightsquigarrow \mathcal{C}(G)$ Cayley graph.
$\chi\colon G \to \mathbb{R}$ homomorphism $\rightsquigarrow \mathcal{C}_\chi(G)$ induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

Definition

$\Sigma^1(G) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}$

An open, conical subset of $\operatorname{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$, independent of choice of generating set for G.

Definition

$\Sigma^k(G, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}$

Here, G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$.

- The BNSR invariants $\Sigma^q(G,\mathbb{Z})$ form a descending chain of *open* subsets of $\operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}$.
- $\Sigma^k(G,\mathbb{Z}) \neq \emptyset \implies G$ is of type FP_k.
- $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ with G/N is abelian:

$$N \text{ is of type } \mathrm{FP}_k \Longleftrightarrow S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$

  where $S(G,N) = \{\chi \in \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \chi(N) = 0\}$.
- In particular:

$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \Longleftrightarrow \{\pm\chi\} \subseteq \Sigma^1(G)$$

Let X be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

Definition

The *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\widehat{\mathbb{Z}G}_{\chi} = \left\{ \lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

$\widehat{\mathbb{Z}G}_{\chi}$ is a ring, contains $\mathbb{Z}G$ as a subring $\implies$ $\widehat{\mathbb{Z}G}_{\chi}$ is a $\mathbb{Z}G$-module.

Definition

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0, \ \forall\, i \leq q\}$$

Bieri: G of type $\mathrm{FP}_k \implies \Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z}), \forall q \leq k$.

Theorem (P.–S.)

If X has finite k-skeleton, then, for every $q \leq k$, then each $\Sigma^q(X, \mathbb{Z})$ is contained in the complement of a union of rationally defined subspaces (explicitly computable).

Corollary

Suppose G is a 1-formal group. Then $\Sigma^1(G) \subseteq \mathcal{R}^1_1(G, \mathbb{R})^{\complement}$.
In particular, if $\mathcal{R}^1_1(G, \mathbb{R}) = H^1(G, \mathbb{R})$, then $\Sigma^1(G) = \emptyset$.

Example

The above inclusion may be strict: Let $G = \langle a, b \mid aba^{-1} = b^2 \rangle$.
Then G is 1-formal, $\Sigma^1(G) = (-\infty, 0)$, yet $\mathcal{R}^1_1(G, \mathbb{R}) = \{0\}$.

Theorem (P.-S.)

$$\Sigma^k(T_L,\mathbb{Z}) \subseteq \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_L,\mathbb{R})\Big)^{\complement}$$

The BNSR invariants of right-angled Artin groups were computed by Meier, Meinert, VanWyk (1998). Comparing their answer with our computation of the resonance varieties, we get:

Corollary (P.-S.)

Suppose $\forall\, \sigma \in \Delta = \Delta_\Gamma$*, and* $\forall\, \mathsf{W} \subseteq \mathsf{V}$ *such that* $\sigma \cap W = \emptyset$*, the groups* $\widetilde{H}_j(\mathrm{lk}_{\Delta_{\mathsf{W}}}(\sigma),\mathbb{Z})$ *are torsion-free,* $\forall\, j \le k - \dim(\sigma) - 2$*. Then:*

$$\Sigma^k(G_\Gamma,\mathbb{Z}) = \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_{\Delta_\Gamma},\mathbb{R})\Big)^{\complement}$$

In particular, for all graphs Γ,

$$\Sigma^1(G_\Gamma,\mathbb{Z}) = \mathcal{R}^1_1(G_\Gamma,\mathbb{R})^{\complement}$$

References

- G. Denham, A. Suciu, *Moment-angle complexes, monomial ideals, and Massey products*, Pure Appl. Math. Quarterly **3** (2007), no. 1, 25–60.
- A. Dimca, S. Papadima, A. Suciu, *Quasi-Kähler Bestvina-Brady groups*, J. Algebraic Geom. **17** (2008), no. 1, 185–197.
- ____, *Topology and geometry of cohomology jump loci*, Duke Math. Journal **148** (2009), no. 3, 405–457.
- S. Papadima, A. Suciu, *Algebraic invariants for right-angled Artin groups*, Math. Annalen **334** (2006), no. 3, 533–555.
- ____, *Algebraic invariants for Bestvina-Brady groups*, J. London Math. Soc. **76** (2007), no. 2, 273–292.
- ____, *Toric complexes and Artin kernels*, Adv. Math. **220** (2009), no. 2, 441–477.
- ____, *Bieri-Neumann-Strebel-Renz invariants and homology jumping loci*, Proc. London Math. Soc. **100** (2010), no. 3, 795–834.
- A. Suciu, *Fundamental groups, Alexander invariants, and cohomology jumping loci*, in: *Topology of algebraic varieties and singularities*, 179–223, Contemp. Math., vol. 538, Amer. Math. Soc., Providence, RI, 2011.