Homological finiteness in the Andreadakis–Johnson filtration

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Reference

Stefan Papadima and Alexander I. Suciu, *Homological finiteness in the Johnson filtration of the automorphism group of a free group*, arxiv:1011.5292

1. The Johnson filtration
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Filtrations and graded Lie algebras

Let G be a group, with commutator $(x, y) = xyx^{-1}y^{-1}$.
Suppose given a descending filtration

$$G = \Phi^1 \supseteq \Phi^2 \supseteq \cdots \supseteq \Phi^s \supseteq \cdots$$

by subgroups of G, satisfying

$$(\Phi^s, \Phi^t) \subseteq \Phi^{s+t}, \quad \forall s, t \geq 1.$$

Then $\Phi^s \triangleleft G$, and Φ^s/Φ^{s+1} is abelian. Set

$$\operatorname{gr}_\Phi(G) = \bigoplus_{s\geq 1} \Phi^s/\Phi^{s+1}.$$

This is a graded Lie algebra, with bracket $[\ ,\ ]\colon \operatorname{gr}^s_\Phi \times \operatorname{gr}^t_\Phi \to \operatorname{gr}^{s+t}_\Phi$ induced by the group commutator.

Basic example: the *lower central series*, $\Gamma^s = \Gamma^s(G)$, defined as

$$\Gamma^1 = G, \Gamma^2 = G', \dots, \Gamma^{s+1} = (\Gamma^s, G), \dots$$

Then for any filtration Φ as above, $\Gamma^s \subseteq \Phi^s$; thus, we have a morphism of graded Lie algebras,

$$\iota_\Phi \colon \operatorname{gr}_\Gamma(G) \longrightarrow \operatorname{gr}_\Phi(G) .$$

Example (P. Hall, E. Witt, W. Magnus)

Let $F_n = \langle x_1, \dots, x_n \rangle$ be the free group of rank n. Then:

- F_n is residually nilpotent, i.e., $\bigcap_{s \geq 1} \Gamma^s(F_n) = \{1\}$.
- $\operatorname{gr}_\Gamma(F_n)$ is isomorphic to the free Lie algebra $\mathcal{L}_n = \operatorname{Lie}(\mathbb{Z}^n)$.
- $\operatorname{gr}^s_\Gamma(F_n)$ is free abelian, of rank $\frac{1}{s} \sum_{d \mid s} \mu(d) n^{\frac{s}{d}}$.
- If $n \geq 2$, the center of $\mathcal{L}_n$ is trivial.

Automorphism groups

Let $\mathrm{Aut}(G)$ be the group of all automorphisms $\alpha\colon G \to G$, with $\alpha \cdot \beta := \alpha \circ \beta$. The *Johnson filtration*,

$$\mathrm{Aut}(G) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^s \supseteq \cdots$$

with terms $F^s = F^s(\mathrm{Aut}(G))$ consisting of those automorphisms which act as the identity on the s-th nilpotent quotient of G:

$$\begin{aligned} F^s &= \ker\big(\mathrm{Aut}(G) \to \mathrm{Aut}(G/\Gamma^{s+1}\big) \\ &= \{\alpha \in \mathrm{Aut}(G) \mid \alpha(x) \cdot x^{-1} \in \Gamma^{s+1},\ \forall x \in G\} \end{aligned}$$

Kaloujnine [1950]: $(F^s, F^t) \subseteq F^{s+t}$.

First term is the *Torelli group*,

$$\mathcal{T}_G = F^1 = \ker\big(\mathrm{Aut}(G) \to \mathrm{Aut}(G_{\mathrm{ab}})\big).$$

By construction, $F^1 = \mathcal{T}_G$ is a normal subgroup of $F^0 = \mathrm{Aut}(G)$. The quotient group,

$$\mathcal{A}(G) = F^0/F^1 = \mathrm{im}(\mathrm{Aut}(G) \to \mathrm{Aut}(G_{\mathrm{ab}}))$$

is the *symmetry group* of $\mathcal{T}_G$; it fits into exact sequence

$$1 \longrightarrow \mathcal{T}_G \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1 .$$

The Torelli group comes endowed with two filtrations:

- The Johnson filtration $\{F^s(\mathcal{T}_G)\}_{s\geq 1}$, inherited from $\mathrm{Aut}(G)$.
- The lower central series filtration, $\{\Gamma^s(\mathcal{T}_G)\}$.

The respective associated graded Lie algebras, $\mathrm{gr}_F(\mathcal{T}_G)$ and $\mathrm{gr}_\Gamma(\mathcal{T}_G)$, come with natural actions of $\mathcal{A}(G)$, and the morphism

$$\iota_F \colon \mathrm{gr}_\Gamma(\mathcal{T}_G) \to \mathrm{gr}_F(\mathcal{T}_G)$$

is $\mathcal{A}(G)$- equivariant.

Automorphism groups of free groups

- Identify $(F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, and $\mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$. The homomorphism $\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ is onto. Thus, $\mathcal{A}(F_n) = \mathrm{GL}_n(\mathbb{Z})$.
- Denote the Torelli group by $\mathrm{IA}_n = \mathcal{T}_{F_n}$, and the Johnson–Andreadakis filtration by $J_n^s = F^s(\mathrm{Aut}(F_n))$.
- Magnus [1934]: IA_n is generated by the automorphisms

$$\alpha_{ij}: \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_\ell \mapsto x_\ell \end{cases} \qquad \alpha_{ijk}: \begin{cases} x_i \mapsto x_i \cdot (x_j, x_k) \\ x_\ell \mapsto x_\ell \end{cases}$$

  with $1 \leq i \neq j \neq k \leq n$.
- Thus, $\mathrm{IA}_1 = \{1\}$ and $\mathrm{IA}_2 = \mathrm{Inn}(F_2) \cong F_2$ are finitely presented.
- Krstić and McCool [1997]: IA_3 is not finitely presentable.
- It is not known whether IA_n admits a finite presentation for $n \geq 4$.

Nevertheless, IA_n has some interesting finitely presented subgroups:

- The McCool group of “pure symmetric” automorphisms, $\mathrm{P}\Sigma_n$, generated by α_{ij}, $1 \le i \ne j \le n$.
- The “upper triangular" McCool group, $\mathrm{P}\Sigma_n^+$, generated by α_{ij}, $i > j$. Cohen, Pakianathan, Vershinin, and Wu [2008]: $\mathrm{P}\Sigma_n^+ = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, with extensions by IA-automorphisms.
- The pure braid group, P_n, consisting of those automorphisms in $\mathrm{P}\Sigma_n$ that leave the word $x_1 \cdots x_n \in F_n$ invariant. $P_n = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$, with extensions by pure braid automorphisms.
- $\mathrm{P}\Sigma_2^+ \cong P_2 \cong \mathbb{Z}$, $\quad \mathrm{P}\Sigma_3^+ \cong P_3 \cong F_2 \times \mathbb{Z}$.
- Question (CPVW): Is $\mathrm{P}\Sigma_n^+ \cong P_n$, for $n \ge 4$? Bardakov and Mikhailov [2008]: $\mathrm{P}\Sigma_4^+ \not\cong P_4$.

The Johnson homomorphism

Given a graded Lie algebra $\mathfrak{g}$, let

$$Der^s(\mathfrak{g}) = \{\delta\colon \mathfrak{g}^\bullet \to \mathfrak{g}^{\bullet+s} \text{ linear} \mid \delta[x,y] = [\delta x, y] + [x, \delta y], \forall x, y \in \mathfrak{g}\}.$$

Then $\mathrm{Der}(\mathfrak{g}) = \bigoplus_{s\geq 1} \mathrm{Der}^s(\mathfrak{g})$ is a graded Lie algebra, with bracket $[\delta, \delta'] = \delta \circ \delta' - \delta' \circ \delta$.

Theorem

Given a group G, there is a monomorphism of graded Lie algebras,

$$J\colon \mathrm{gr}_F(\mathcal{T}_G) \longrightarrow \mathrm{Der}(\mathrm{gr}_\Gamma(G)),$$

given on homogeneous elements $\alpha \in F^s(\mathcal{T}_G)$ and $x \in \Gamma^t(G)$ by

$$J(\bar{\alpha})(\bar{x}) = \overline{\alpha(x)\cdot x^{-1}}.$$

Moreover, J is equivariant with respect to the natural actions of $\mathcal{A}(G)$.

The Johnson homomorphism informs on the Johnson filtration.

Theorem

Let G be a group. For each $q \geq 1$, the following are equivalent:

1. $J \circ \iota_F \colon \operatorname{gr}^s_\Gamma(\mathcal{T}_G) \to \operatorname{Der}^s(\operatorname{gr}_\Gamma(G))$ *is injective, for all* $s \leq q$.
2. $\Gamma^s(\mathcal{T}_G) = F^s(\mathcal{T}_G)$, *for all* $s \leq q+1$.

Proposition

Suppose G is residually nilpotent, $\operatorname{gr}_\Gamma(G)$ is centerless, and $J \circ \iota_F \colon \operatorname{gr}^1_\Gamma(\mathcal{T}_G) \to \operatorname{Der}^1(\operatorname{gr}_\Gamma(G))$ is injective. Then $F^2(\mathcal{T}_G) = \mathcal{T}'_G$.

Let $\mathrm{Inn}(G) = \mathrm{im}(\mathrm{Ad}\colon G \to \mathrm{Aut}(G))$, where $\mathrm{Ad}_x\colon G \to G$, $y \mapsto xyx^{-1}$. Define the *outer* automorphism group of a group G by

$$1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \xrightarrow{\pi} \mathrm{Out}(G) \longrightarrow 1 .$$

Obtain:

- Filtration $\{\widetilde{F}^s\}_{s\geq 0}$ on $\mathrm{Out}(G)$: $\widetilde{F}^s := \pi(F^s)$.
- The *outer Torelli group* of G: subgroup $\widetilde{\mathcal{T}}_G = \widetilde{F}^1$ of $\mathrm{Out}(G)$
- Exact sequence: $1 \longrightarrow \widetilde{\mathcal{T}}_G \longrightarrow \mathrm{Out}(G) \longrightarrow \mathcal{A}(G) \longrightarrow 1$.

Let $\mathfrak{g}$ be a graded Lie algebra, and $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$, where $\mathrm{ad}_x\colon \mathfrak{g} \to \mathfrak{g}$, $y \mapsto [x, y]$. Define the Lie algebra of outer derivations of $\mathfrak{g}$ by

$$0 \longrightarrow \mathrm{im}(\mathrm{ad}) \longrightarrow \mathrm{Der}(\mathfrak{g}) \xrightarrow{q} \widetilde{\mathrm{Der}}(\mathfrak{g}) \longrightarrow 0 .$$

Theorem

Suppose $Z(\mathrm{gr}_\Gamma(G)) = 0$. Then the Johnson homomorphism induces an $\mathcal{A}(G)$-equivariant monomorphism of graded Lie algebras,

$$\widetilde{J}\colon\ \mathrm{gr}_{\widetilde{F}}(\widetilde{\mathcal{T}}_G) \longrightarrow \widetilde{\mathrm{Der}}(\mathrm{gr}_\Gamma(G))\,.$$

To summarize:

$$\begin{array}{ccccc}
\mathrm{gr}_\Gamma(G) & \xrightarrow{\quad = \quad} & \mathrm{gr}_\Gamma(G) & \xrightarrow{\quad = \quad} & \mathrm{gr}_\Gamma(G) \\
\Big\downarrow{\scriptstyle \mathrm{gr}_\Gamma(\mathrm{Ad})} & & \Big\downarrow{\scriptstyle \overline{\mathrm{Ad}}} & & \Big\downarrow{\scriptstyle \mathrm{ad}} \\
\mathrm{gr}_\Gamma(\mathcal{T}_G) & \xrightarrow{\ \iota_F\ } & \mathrm{gr}_F(\mathcal{T}_G) & \xhookrightarrow{\quad J \quad} & \mathrm{Der}(\mathrm{gr}_\Gamma(G)) \\
\Big\downarrow{\scriptstyle \mathrm{gr}_\Gamma(\pi)} & & \Big\downarrow{\scriptstyle \bar{\pi}} & & \Big\downarrow{\scriptstyle q} \\
\mathrm{gr}_\Gamma(\widetilde{\mathcal{T}}_G) & \xrightarrow{\ \iota_{\widetilde{F}}\ } & \mathrm{gr}_{\widetilde{F}}(\widetilde{\mathcal{T}}_G) & \xhookrightarrow{\quad \widetilde{J} \quad} & \widetilde{\mathrm{Der}}(\mathrm{gr}_\Gamma(G))\,,
\end{array}$$

The Torelli group of F_n

Let $\mathcal{T}_{F_n} = J_n^1 = IA_n$ be the Torelli group of F_n. Recall we have an equivariant $\mathrm{GL}_n(\mathbb{Z})$-homomorphism,

$$J\colon \operatorname{gr}_F(\mathrm{IA}_n) \to \operatorname{Der}(\mathcal{L}_n),$$

In degree 1, this can be written as

$$J\colon \operatorname{gr}_F^1(\mathrm{IA}_n) \to H^* \otimes (H \wedge H),$$

where $H = (F_n)_{\mathrm{ab}} = \mathbb{Z}^n$, viewed as a $\mathrm{GL}_n(\mathbb{Z})$-module via the defining representation. Composing with ι_F, we get a homomorphism

$$J \circ \iota_F \colon (\mathrm{IA}_n)_{\mathrm{ab}} \longrightarrow H^* \otimes (H \wedge H)\ .$$

Theorem (Andreadakis, Cohen–Pakianathan, Farb, Kawazumi)

For each $n \geq 3$, the map $J \circ \iota_F$ is a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism.

Thus, $H_1(\mathrm{IA}_n, \mathbb{Z})$ is free abelian, of rank $b_1(IA_n) = n^2(n-1)/2$.

We have a commuting diagram,

$$\begin{array}{ccccccccc}
 & & \mathrm{Inn}(F_n) & \xrightarrow{=} & \mathrm{Inn}(F_n) & & & & \\
 & & \big\downarrow & & \big\downarrow & & & & \\
1 & \longrightarrow & \mathrm{IA}_n & \longrightarrow & \mathrm{Aut}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle =} & & \\
1 & \longrightarrow & \mathrm{OA}_n & \longrightarrow & \mathrm{Out}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1
\end{array}$$

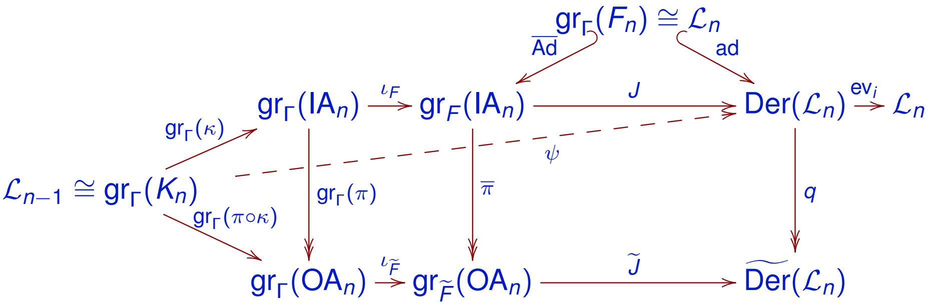
- Thus, $\mathrm{OA}_n = \widetilde{\mathcal{T}}_{F_n}$.
- Write the induced Johnson filtration on $\mathrm{Out}(F_n)$ as $\widetilde{J}_n^s = \pi(J_n^s)$.
- $\mathrm{GL}_n(\mathbb{Z})$ acts on $(\mathrm{OA}_n)_{\mathrm{ab}}$, and the outer Johnson homomorphism defines a $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphism

$$\widetilde{J} \circ \iota_{\widetilde{F}} \colon (\mathrm{OA}_n)_{\mathrm{ab}} \xrightarrow{\cong} H^* \otimes (H \wedge H)/H .$$

- Moreover, $\widetilde{J}_n^2 = \mathrm{OA}_n'$, and we have an exact sequence

$$1 \longrightarrow F_n' \xrightarrow{\mathrm{Ad}} \mathrm{IA}_n' \longrightarrow \mathrm{OA}_n' \longrightarrow 1 .$$

Consider the commuting diagram



where

- $K_n := \ker(\mathrm{P\Sigma}_n^+ \twoheadrightarrow \mathrm{P\Sigma}_{n-1}^+) \cong F_{n-1}$, and $\kappa\colon K_n \hookrightarrow \mathrm{IA}_n$ inclusion.
- $\mathrm{ev}_i\colon \mathrm{Der}^*(\mathcal{L}_n) \to \mathcal{L}_n^{*+1}$, $\mathrm{ev}_i(\delta) = \delta(\bar{x}_i)$.

Then, the restriction of $\mathrm{ev}_i \circ \psi$ to $\mathcal{L}_{n-1}^s$ equals $(-1)^s \mathrm{ad}_{\bar{x}_n}$ if $i = n$, and 0 otherwise. Moreover, $\mathrm{im}(\psi) \cap \mathrm{im}(\mathrm{ad}) = \{0\}$.

From this, we get:

Theorem

Let G be either IA_n or OA_n, and assume $n \geq 3$. Then:

1. *The $\mathbb{Q}$-vector space $\mathrm{gr}_\Gamma(G) \otimes \mathbb{Q}$ is infinite-dimensional.*
2. *The $\mathbb{Q}G_{\mathrm{ab}}$-module $H_1(G', \mathbb{Q})$ is* not *trivial.*

Deeper into the Johnson filtration

Conjecture (F. Cohen, A. Heap, A. Pettet 2010)

If $n \geq 3$, $s \geq 2$, and $1 \leq i \leq n-2$, the cohomology group $H^i(J_n^s, \mathbb{Z})$ is not finitely generated.

We disprove this conjecture, at least rationally, in the case when $n \geq 5$, $s = 2$, and $i = 1$.

Theorem

If $n \geq 5$, then $\dim_{\mathbb{Q}} H^1(J_n^2, \mathbb{Q}) < \infty$.

To start with, note that $J_n^2 = IA_n'$. Thus, it remains to prove that $b_1(\mathrm{IA}_n') < \infty$, i.e., $(IA_n'/IA_n'') \otimes \mathbb{Q}$ is finite dimensional.

The Alexander invariant

Let G be a group. Recall $G' = (G, G)$ and $G_{ab} = G/G'$ is the maximal abelian quotient of G.

Similarly, $G'' = (G', G')$ and G/G'' is the maximal metabelian quotient.

Get exact sequence $0 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G_{ab} \longrightarrow 0$.

Conjugation in G/G'' turns the abelian group

$$B(G) := G'/G'' = H_1(G', \mathbb{Z})$$

into a module over $R = \mathbb{Z}G_{ab}$, called the *Alexander invariant* of G.

Since both G' and G'' are characteristic subgroups of G, the action of $\mathrm{Aut}(G)$ on G induces an action on $B(G)$. Although this action need not respect the R-module structure, we have:

Proposition

The Torelli group $\mathcal{T}_G$ acts R-linearly on the Alexander invariant $B(G)$.

Characteristic varieties

Let G be a finitely generated group.

- The *character group* $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ is an algebraic group.
- The projection $\mathrm{ab}\colon G \to G_{\mathrm{ab}}$ induces an isomorphism $\widehat{G}_{\mathrm{ab}} \xrightarrow{\simeq} \widehat{G}$.
- The identity component, $\widehat{G}^0$, is isomorphic to a complex algebraic torus of dimension $n = \mathrm{rank}\, G_{\mathrm{ab}}$.
- The coordinate ring of $\widehat{G} = H^1(G, \mathbb{C}^\times)$ is $R_{\mathbb{C}} = \mathbb{C}[G_{\mathrm{ab}}]$.
- The (first) *characteristic variety* of G is the support of the Alexander invariant: $\mathcal{V}(G) = V(\mathrm{ann}\, B) \cup \{1\} \subset \widehat{G}$.
- $\mathcal{V}(G)$ finite $\Longleftrightarrow \dim_{\mathbb{Q}} B(G) \otimes \mathbb{Q} < \infty$.

Example

If $G = \mathbb{Z}^n$, then $B(G) = 0$ and $\mathcal{V}(G) = \{1\} \subset (\mathbb{C}^\times)^n$.
If $G = F_n$, $n \geq 2$, then $\mathcal{V}(G) = (\mathbb{C}^\times)^n$.

Resonance varieties

- Let $\cup \colon H^1(G,\mathbb{C}) \wedge H^1(G,\mathbb{C}) \to H^2(G,\mathbb{C})$ be the cup-product map.
- The (first) *resonance variety* of G is defined as

$$\mathcal{R}(G) = \{z \in H^1(G,\mathbb{C}) \mid \exists u \in H^1(G,\mathbb{C}), u \neq \lambda z \text{ and } z \cup u = 0\}.$$

- This is a homogeneous algebraic subvariety of $H^1(G,\mathbb{C}) = \mathbb{C}^n$, where $n = b_1(G)$.
- Let $\mathrm{TC}_1(\mathcal{V}(G))$ be the tangent cone to $\mathcal{V}(G)$ at 1, viewed as a subset of $T_1(\mathbb{T}(G)) = H^1(G,\mathbb{C})$. Then:

$$\mathrm{TC}_1(\mathcal{V}(G)) \subseteq \mathcal{R}(G).$$

Example

If $G = \mathbb{Z}^n$, then $\mathcal{R}(G) = \{0\}$.
If $G = F_n$, $n \geq 2$, then $\mathcal{R}(G) = \mathbb{C}^n$.

Representations of $\mathfrak{sl}_n(\mathbb{C})$

- $\mathfrak{h}$: the Cartan subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, with coordinates $t_1, \dots, t_n$.
- $\{t_i - t_j \mid 1 \le i < j \le n\}$: the positive roots of $\mathfrak{sl}_n(\mathbb{C})$.
- $\lambda_i = t_1 + \cdots + t_i$.
- $V(\lambda)$: the irreducible, finite dimensional representation of $\mathfrak{sl}_n(\mathbb{C})$ with highest weight $\lambda = \sum_{i<n} a_i\lambda_i$, with $a_i \in \mathbb{Z}_{\ge 0}$.

Set $H_{\mathbb{C}} = H_1(F_n, \mathbb{C}) = \mathbb{C}^n$, and

$$V := H^1(\mathrm{OA}_n, \mathbb{C}) = H_{\mathbb{C}} \otimes (H_{\mathbb{C}}^* \wedge H_{\mathbb{C}}^*)/H_{\mathbb{C}}^*.$$

$$K := \ker\big(\cup\colon V \wedge V \to H^2(\mathrm{OA}_n, \mathbb{C})\big).$$

Theorem (Pettet 2005)

Fix $n \ge 4$, and set $\lambda = \lambda_1 + \lambda_{n-2}$ and $\mu = \lambda_1 + \lambda_{n-2} + \lambda_{n-1}$ Then $V = V(\lambda)$ and $K = V(\mu)$, as $\mathfrak{sl}_n(\mathbb{C})$-modules.

Theorem

$\mathcal{R}(\mathrm{OA}_n) = \{0\}$, *for all* $n \geq 4$.

Proof.

- Let $u_0 \in V(\mu)$ be a maximal vector.
- Suppose $\mathcal{R} \neq \{0\}$. Then, since $\mathcal{R}$ is a Zariski closed, $\mathfrak{sl}_n(\mathbb{C})$-invariant cone in $V(\lambda)$, it must contain a maximal vector $v_0 \in V(\lambda)$. (This follows from the Borel fixed point theorem.)
- Since $v_0 \in \mathcal{R}$, there is a $w \in V(\lambda)$ such that $u_0 = v_0 \wedge w$.
- Let $x \in \mathfrak{sl}_n(\mathbb{C})^+$. Since u_0, v_0 are max vectors, $xu_0 = xv_0 = 0$.
- Since $u_0 = v_0 \wedge w$, we have $xu_0 = xv_0 \wedge w + v_0 \wedge xw$.
- Hence, $v_0 \wedge xw = 0$, and thus $xw \in \mathbb{C} \cdot v_0$.
- This implies $w = 0$, and so $u_0 = v_0 \wedge w = 0$, contradicting the maximality of u_0.

□

Let S be a complex, simple linear algebraic group defined over $\mathbb{Q}$, with $\mathbb{Q}$-rank$(S) \geq 1$, and let Γ be an arithmetic subgroup of S.

Theorem (Dimca, Papadima 2010)

Suppose Γ acts on a lattice L, such that the action of Γ on $L \otimes \mathbb{C}$ extends to a rational, irreducible S-representation. Then, the corresponding action of Γ on the complex algebraic torus $\widehat{L} = \mathrm{Hom}(L, \mathbb{C}^{\times})$ is geometrically irreducible*, i.e., the only Γ-invariant, Zariski closed subsets of $\widehat{L}$ are either equal to $\widehat{L}$, or finite.*

Theorem

If $n \geq 4$, then $\mathcal{V}(\mathrm{OA}_n)$ is finite, and so $b_1(\mathrm{OA}'_n) < \infty$.

Proof.

- Set $S = \mathfrak{sl}_n(\mathbb{C})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $L = (\mathrm{OA}_n)_{\mathrm{ab}}$. By above result: $\widehat{L} = H^1(\mathrm{OA}_n, \mathbb{C}^\times)$ is geometrically Γ-irreducible.
- The variety $\mathcal{V} = \mathcal{V}(\mathrm{OA}_n)$ is a Γ-invariant, Zariski closed subset of $\widehat{L}$.
- Suppose $\mathcal{V}$ is infinite. Then $\mathcal{V} = \widehat{L}$, and so $\mathcal{R}(\mathrm{OA}_n) = H^1(\mathrm{OA}_n, \mathbb{C})$, contradicting $\mathcal{R} = \{0\}$.

□

Theorem

If $n \geq 5$*, then* $b_1(\mathrm{IA}'_n) < \infty$.

Proof.

For each n, the Hochschild-Serre spectral sequence of the extension $1 \longrightarrow F'_n \longrightarrow \mathrm{IA}'_n \longrightarrow \mathrm{OA}'_n \longrightarrow 1$ gives rise to exact sequence

$$H_1(F'_n, \mathbb{C})_{\mathrm{IA}'_n} \longrightarrow H_1(\mathrm{IA}'_n, \mathbb{C}) \longrightarrow H_1(\mathrm{OA}'_n, \mathbb{C}) \longrightarrow 0 .$$

The last term is finite-dimensional for all $n \geq 4$ by previous theorem, while the first term is finite-dimensional for all $n \geq 5$, by the nilpotency of the action of IA'_n on F'_n/F''_n.

□