The pure braid groups and their relatives

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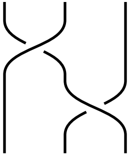


- Let B_n be the group of braids on n strings (under concatenation).
- B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.
- Let $P_n = \ker(B_n \twoheadrightarrow S_n)$ be the pure braid group on n strings.
- P_n is generated by $A_{ij} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$ for $1 \leqslant i < j \leqslant n$.

- $B_n = \mathrm{Mod}^1_{0,n}$, the mapping class group of the 2-disk with n marked points.
- Thus, B_n is a subgroup of $\mathrm{Aut}(F_n)$, and $P_n \subset \mathrm{IA}_n$. In fact:

$$B_n = \{\beta \in \mathrm{Aut}(F_n) \mid \beta(x_i) = wx_{\tau(i)}w^{-1}, \beta(x_1 \cdots x_n) = x_1 \cdots x_n\}.$$

- A classifying space for P_n is the configuration space

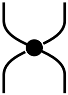
$$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

- Thus, $B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C})/S_n)$.
- Moreover,

$$P_n = F_{n-1} \rtimes_{\alpha_{n-1}} P_{n-1} = F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1,$$

where $\alpha_n \colon P_n \subset B_n \hookrightarrow \mathrm{Aut}(F_n)$.

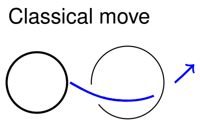
WELDED BRAID GROUPS



- The set of all permutation-conjugacy automorphisms of F_n forms a subgroup of $wB_n \subset \operatorname{Aut}(F_n)$, called the welded braid group.
- Let $wP_n = \ker(wB_n \twoheadrightarrow S_n) = \mathrm{IA}_n \cap wB_n$ be the pure welded braid group wP_n.
- McCool (1986) gave a finite presentation for wP_n. It is generated by the automorphisms α_{ij} ($1 \leqslant i \neq j \leqslant n$) sending $x_i \mapsto x_j x_i x_j^{-1}$ and $x_k \mapsto x_k$ for $k \neq i$, subject to the relations

$$\begin{aligned} \alpha_{ij}\alpha_{ik}\alpha_{jk} &= \alpha_{jk}\alpha_{ik}\alpha_{ij} && \text{for } i,j,k \text{ distinct,}\\ [\alpha_{ij},\alpha_{st}] &= 1 && \text{for } i,j,s,t \text{ distinct,}\\ [\alpha_{ik},\alpha_{jk}] &= 1 && \text{for } i,j,k \text{ distinct.} \end{aligned}$$

- The group wB_n (respectively, wP_n) is the fundamental group of the space of untwisted flying rings (of unequal diameters), cf. Brendle and Hatcher (2013).



- The upper pure welded braid group (or, upper McCool group) is the subgroup $wP_n^+ \subset wP_n$ generated by α_{ij} for $i < j$.
- We have $wP_n^+ \cong F_{n-1} \rtimes \cdots \rtimes F_2 \rtimes F_1$.

PROPOSITION (S.–WANG)

For $n \geqslant 4$, the inclusion $wP_n^+ \hookrightarrow wP_n$ admits no splitting.

Virtual braid groups

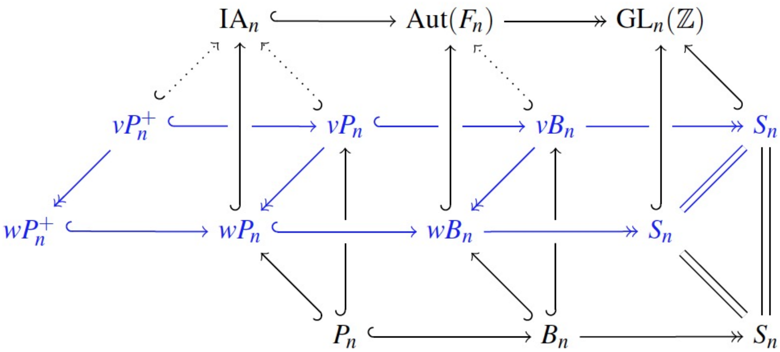
- The virtual braid group vB_n is obtained from wB_n by omitting certain commutation relations.
- Let $vP_n = \ker(vB_n \to S_n)$ be the pure virtual braid group.
- Bardakov (2004) gave a presentation for vP_n, with generators x_{ij} for $1 \leqslant i \neq j \leqslant n$,

subject to the relations

$$x_{ij}x_{ik}x_{jk} = x_{jk}x_{ik}x_{ij}, \qquad \text{for } i, j, k \text{ distinct,}$$
$$[x_{ij}, x_{st}] = 1, \qquad \text{for } i, j, s, t \text{ distinct.}$$

- Let vP_n^+ be the subgroup of vP_n generated by x_{ij} for $i < j$.
- The inclusion $vP_n^+ \hookrightarrow vP_n$ is a split injection.
- Bartholdi, Enriquez, Etingof, and Rains (2006) studied vP_n and vP_n^+ as groups arising from the Yang-Baxter equation.
- They constructed classifying spaces for these groups by taking quotients of permutahedra by suitable actions of the symmetric groups.

SUMMARY OF BRAID-LIKE GROUPS



Cohomology rings and Betti numbers

The cohomology algebras of the pure-braid like groups:

- $H^*(P_n, \mathbb{C})$: Arnol'd (1969).
- $H^*(wP_n, \mathbb{C})$: Jensen, McCammond, and Meier (2006).
- $H^*(wP_n^+; \mathbb{C})$: F. Cohen, Pakhianathan, Vershinin, and Wu (2007) .
- $H^*(vP_n; \mathbb{C})$ and $H^*(vP_n^+; \mathbb{C})$: Bartholdi et al (2006), P. Lee (2013).

The Betti numbers of the pure-braid like groups:

	P_n	wP_n	wP_n^+	vP_n	vP_n^+
b_i	$s(n, n-i)$	$\binom{n-1}{i}n^i$	$s(n, n-i)$	$L(n, n-i)$	$S(n, n-i)$

Here $s(n, k)$ are the Stirling numbers of the first kind, $S(n, k)$ are the Stirling numbers of the second kind, and $L(n, k)$ are the Lah numbers.

	$H^*(P_n;\mathbb{C})$	$H^*(wP_n;\mathbb{C})$	$H^*(wP_n^+;\mathbb{C})$	$H^*(vP_n;\mathbb{C})$	$H^*(vP_n^+;\mathbb{C})$
Generators	u_{ij} $(i<j)$	a_{ij} $(i\neq j)$	e_{ij} $(i<j)$	a_{ij} $(i\neq j)$	e_{ij} $(i<j)$
Relations	(I1)	(I2) (I3)	(I5)	(I2)(I3)(I4)	(I5) (I6)
Koszul	Yes	No for $n\geqslant 4$	Yes	Yes	Yes

$$\begin{array}{lll}
\text{(I1)} & u_{jk}u_{ik} = u_{ij}(u_{ik}-u_{jk}) & \text{for } i<j<k,\\
\text{(I2)} & a_{ij}a_{ji} = 0 & \text{for } i\neq j,\\
\text{(I3)} & a_{kj}a_{ik} = a_{ij}(a_{ik}-a_{jk}) & \text{for } i,j,k \text{ distinct},\\
\text{(I4)} & a_{ji}a_{ik} = (a_{ij}-a_{ik})a_{jk} & \text{for } i,j,k \text{ distinct},\\
\text{(I5)} & e_{ij}(e_{ik}-e_{jk}) = 0 & \text{for } i<j<k,\\
\text{(I6)} & (e_{ij}-e_{ik})e_{jk} = 0 & \text{for } i<j<k.
\end{array}$$

- Koszulness for P_n: Arnol'd, Kohno.
- Koszulness for vP_n and vP_n^+: Bartholdi et al (2006), Lee (2013).
- Koszulness for wP_n^+: D. Cohen and G. Pruidze (2008).
- Non-Koszulness for wP_n: Conner and Goetz (2015).

ASSOCIATED GRADED LIE ALGEBRAS

For a finitely generated group G, define the *lower central series* inductively by $\gamma_1 G = G$ and $\gamma_{k+1} G = [\gamma_k G, G]$.
The group commutator induces a graded Lie algebra structure on

$$\mathrm{gr}(G) = \bigoplus_{k\geqslant 1} (\gamma_k G/\gamma_{k+1} G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

	$\mathrm{gr}(P_n)$	$\mathrm{gr}(wP_n)$	$\mathrm{gr}(wP_n^+)$	$\mathrm{gr}(vP_n)$	$\mathrm{gr}(vP_n^+)$
Generators	x_{ij}, $i<j$	x_{ij}, $i \neq j$	x_{ij}, $i<j$	x_{ij}, $i \neq j$	x_{ij}, $i<j$
Relations	L2, L4	L1, L2, L3	L1, L2, L3	L1, L2	L1, L2
	Kohno, Falk–Randell	Jensen et al.	F. Cohen et al.	Bartholdi et al., Lee	Bartholdi et al., Lee

(L1) $[x_{ij}, x_{ik}] + [x_{ij}, x_{jk}] + [x_{ik}, x_{jk}] = 0$ for distinct i, j, k,

(L2) $[x_{ij}, x_{kl}] = 0$ for $\{i, j\} \cap \{k, l\} = \varnothing$,

(L3) $[x_{ik}, x_{jk}] = 0$ for distinct i, j, k,

(L4) $[x_{im}, x_{ij} + x_{ik} + x_{jk}] = 0$ for $m = j, k$ and i, j, m distinct.

- Let $\phi_k(G) = \dim \mathrm{gr}_k(G)$ be the *LCS ranks* of G.
- E.g.: $\phi_k(F_n) = \frac{1}{k}\sum_{d|k} \mu(\frac{k}{d}) n^d$.
- By the Poincaré–Birkhoff–Witt theorem,

$$\prod_{k=1}^{\infty}(1-t^k)^{-\phi_k(G)} = \mathrm{Hilb}(U(\mathrm{gr}(G)), t).$$

Proposition (Papadima–Yuzvinsky 1999)

Suppose $\mathrm{gr}(G)$ is quadratic and $A = H^*(G;\mathbb{C})$ is Koszul. Then $\mathrm{Hilb}(U(\mathrm{gr}(G)), t) \cdot \mathrm{Hilb}(A, -t) = 1$.

- If G is a pure braid-like group, then $\mathrm{gr}(G)$ is quadratic.
- Furthermore, if $G \neq wP_n$ ($n \geqslant 4$), then $H^*(G;\mathbb{C})$ is Koszul.
- Thus,

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k(G)} = \sum_{i\geqslant 0} b_i(G)(-t)^i.$$

CHEN LIE ALGEBRAS

- The *Chen Lie algebra* of a f.g. group G is $\mathrm{gr}(G/G'')$, the associated graded Lie algebra of its maximal metabelian quotient.
- Let $\theta_k(G) = \dim \mathrm{gr}_k(G/G'')$ be the *Chen ranks* of G.
- Easy to see: $\theta_k(G) \leqslant \phi_k(G)$ and $\theta_k(G) = \phi_k(G)$ for $k \leqslant 3$.
- Chen(1951): $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$ for $k \geqslant 2$.

THEOREM (D. COHEN–S. 1993)

The Chen ranks $\theta_k = \theta_k(P_n)$ are given by $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, and $\theta_k = (k-1)\binom{n+1}{4}$ for $k \geqslant 3$.

COROLLARY

Let $\Pi_n = F_{n-1} \times \cdots \times F_1$. Then $P_n \not\cong \Pi_n$ for $n \geqslant 4$, although both groups have the same Betti numbers and LCS ranks.

THEOREM (D. COHEN–SCHENCK 2015)

$\theta_k(wP_n) = (k-1)\binom{n}{2} + (k^2-1)\binom{n}{3}$, *for* $k \gg 0$.

THEOREM (S.–WANG)

The Chen ranks $\theta_k = \theta_k(wP_n^+)$ *are given by* $\theta_1 = \binom{n}{2}$, $\theta_2 = \binom{n}{3}$, *and*

$$\theta_k = \sum_{i=3}^{k} \binom{n+i-2}{i+1} + \binom{n+1}{4}, \quad \text{for } k \geqslant 3.$$

COROLLARY

$wP_n^+ \not\cong P_n$ *and* $wP_n^+ \not\cong \Pi_n$ *for* $n \geqslant 4$, *although all three groups have the same Betti numbers and LCS ranks.*

This answers a question of F. Cohen et al. (2007).
For $n = 4$, an incomplete argument was given by Bardakov and Mikhailov (2008), using single-variable Alexander polynomials.

RESONANCE VARIETIES

- Let G be a finitely presented group, and set $A = H^*(G, \mathbb{C})$.
- The (first) *resonance variety* of G is given by

$$\mathcal{R}_1(G) = \{a \in A^1 \mid \exists b \in A^1 \backslash \mathbb{C} \cdot a \text{ such that } a \cdot b = 0 \in A^2\}.$$

- For instance, $\mathcal{R}_1(F_n) = \mathbb{C}^n$ for $n \geqslant 2$, and $\mathcal{R}_1(\mathbb{Z}^n) = \{0\}$.

THEOREM (D. COHEN–S. 1999)

$\mathcal{R}_1(P_n)$ *is a union of* $\binom{n}{3} + \binom{n}{4}$ *linear subspaces of dimension* 2.

THEOREM (D. COHEN 2009)

$\mathcal{R}_1(wP_n)$ *is a union of* $\binom{n}{2}$ *linear subspaces of dimension* 2 *and* $\binom{n}{3}$ *linear subspaces of dimension* 3.

Theorem (S.–Wang)

$$\mathcal{R}_1(wP_n^+) = \bigcup_{2\leqslant i<j\leqslant n} L_{ij},$$

where L_{ij} is a linear subspace of dimension i.

Proposition (Bardakov–Mikhailov–Vershinin–Wu 2009, S.–Wang)

$\mathcal{R}_1(vP_3)$ *coincides with* $H^1(vP_3, \mathbb{C}) = \mathbb{C}^6$.

Proposition (S.–Wang)

$\mathcal{R}_1(vP_4^+)$ *is the subvariety of* $H^1(vP_4^+, \mathbb{C}) = \mathbb{C}^6$ *defined by*

$$x_{12}x_{24}(x_{13}+x_{23}) + x_{13}x_{34}(x_{12}-x_{23}) - x_{24}x_{34}(x_{12}+x_{13}) = 0,$$
$$x_{12}x_{23}(x_{14}+x_{24}) + x_{12}x_{34}(x_{23}-x_{14}) + x_{14}x_{34}(x_{23}+x_{24}) = 0,$$
$$x_{13}x_{23}(x_{14}+x_{24}) + x_{14}x_{24}(x_{13}+x_{23}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0,$$
$$x_{12}(x_{13}x_{14}-x_{23}x_{24}) + x_{34}(x_{13}x_{23}-x_{14}x_{24}) = 0.$$

FORMALITY PROPERTIES

- (Quillen 1968) The *Malcev Lie algebra* of a group G is
$$\mathfrak{m}(G) = \mathrm{Prim}(\widehat{\mathbb{C}G}),$$
the primitives in the I-adic completion of the group algebra of G.
- A complete, filtered Lie algebra with $\mathrm{gr}(\mathfrak{m}(G)) \cong \mathrm{gr}(G)$.
- A f.g. group G is 1*-formal* if its Malcev Lie algebra is quadratic.
- Thus, if G is 1-formal, then G is *graded-formal*, i.e., $\mathrm{gr}(G)$ is quadratic.
- Conversely, if G is graded-formal and *filtered-formal*, then G is 1-formal.
- Formality properties are preserved under (finite) direct products and free products.

THEOREM (DIMCA–PAPADIMA–S. 2009)

If G is 1-formal, then $\mathcal{R}_1(G)$ is a union of projectively disjoint, rationally defined linear subspaces of $H^1(G, \mathbb{C})$.

THEOREM (KOHNO 1983)

Fundamental groups of complements of complex projective hypersurfaces (e.g., F_n and P_n) are 1-formal.

THEOREM (BERCEANU–PAPADIMA 2009)

wP_n and wP_n^+ are 1-formal.

THEOREM (S.–WANG)

vP_n and vP_n^+ are 1-formal if and only if $n \leqslant 3$.

PROOF.

- There are split monomorphisms

$$\begin{array}{ccccccccccc}
vP_2^+ & \hookrightarrow & vP_3^+ & \hookrightarrow & vP_4^+ & \hookrightarrow & vP_5^+ & \hookrightarrow & vP_6^+ & \hookrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
vP_2 & \hookrightarrow & vP_3 & \hookrightarrow & vP_4 & \hookrightarrow & vP_5 & \hookrightarrow & vP_6 & \hookrightarrow & \dots
\end{array}$$

- $vP_2^+ = \mathbb{Z}$ and $vP_3^+ \cong \mathbb{Z} * \mathbb{Z}^2$. Thus, they are both 1-formal.
- $vP_3 \cong N * \mathbb{Z}$ and $P_4 \cong N \times \mathbb{Z}$. Thus, vP_3 is 1-formal.
- $\mathcal{R}_1(vP_4^+)$ is non-linear. Thus, vP_4^+ is not 1-formal.
- Hence, vP_n^+ and vP_n ($n \geqslant 4$) are also not 1-formal.

□

THE CHEN RANKS CONJECTURE

CONJECTURE (S. 2001)

Let G be a hyperplane arrangement group. Let c_r be the number of r-dimensional components of $\mathcal{R}_1(G)$. Then, for $k \gg 1$,

$$\theta_k(G) = \sum_{r\geqslant 2} c_r \cdot \theta_k(F_r).$$

The conjecture was based in part on $\theta_k(P_n)$ versus $\mathcal{R}_1(P_n)$.

THEOREM (D. COHEN–SCHENCK 2014)

More generally, the conjecture holds if G is a 1-formal, commutator-relators group for which $\mathcal{R}_1(G)$ is 0-isotropic, projectively disjoint, and reduced as a scheme.

- The groups wP_n satisfy the Chen ranks formula.
- However, wP_n^+ do *not* satisfy the Chen ranks formula for $n \geqslant 4$!

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