The work of Ştefan Papadima in topology and geometry

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INTRODUCTION

- The work of Ştefan Papadima spans some four decades (1977–2017).
- His research covered many areas of Algebraic, Geometric, and Differential Topology; Algebraic and Differential Geometry; Several Complex Variables; Group Theory; Lie Algebras; and Combinatorics.

Bucharest 1980

- He published over 70 articles, many in top journals, with half a dozen papers still coming out.
- The two of us collaborated on 28 papers, starting in late 1999 during a 6-week Research in Pairs at Oberwolfach, with the last one completed in November 2017.

Here are some of the themes from Papadima's work:

- Rational Homotopy Theory
  - Rational homotopy of Thom spaces
  - Formality of spaces and maps
  - Rational classification of differentiable manifolds
  - Rigidity properties of homogeneous spaces
  - Isometry-invariant geodesics
  - Closed manifolds and Artinian complete intersections
  - Rational $K(\pi, 1)$ spaces and Koszul algebras
  - Finite algebraic models and actions of compact Lie groups

Boston 2006

- Lie Algebras
  - Malcev Lie algebras
  - Holonomy Lie algebras
  - Chen Lie algebras
  - Homotopy Lie algebras and the Rescaling Formula
  - Infinitesimal finiteness obstructions
- Discrete Groups
  - Braids and Campbell-Hausdorff invariants
  - Finite-type invariants for braid groups
  - Right-angled Artin groups
  - Bestvina–Brady groups
  - McCool groups
  - Finiteness properties for Torelli groups
  - Johnson filtration of automorphism groups

Trieste 2006


Venice 2007

- Hyperplane Arrangements
  - Hypersolvable arrangements
  - Decomposable arrangements
  - Homotopy theory of complements of arrangements
  - Minimality of arrangement complements
  - Milnor fibrations of arrangements

Nice 2009

- Cohomology Jump Loci and Representation Varieties
  - Germs of cohomology jump loci
  - The Tangent Cone Formula
  - Jump loci for quasi-projective manifolds
  - Vanishing resonance and representations of Lie algebras
  - Representation varieties and deformation theory
  - Higher rank cohomology jump loci
  - Naturality properties of embedded jump loci

Associated graded Lie algebras

- Let G be a group. The *lower central series* of G is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G' = [G, G]$, and
$$\gamma_{k+1}(G) = [\gamma_k(G), G].$$
- Then $\gamma_k(G) \lhd G$, and $\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$ is abelian. Set
$$\operatorname{gr}(G) = \bigoplus_{k \geqslant 1} \operatorname{gr}_k(G).$$
- This is a graded Lie algebra, with Lie bracket $[\, ,\,]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.
- If G is finitely generated, then $\operatorname{gr}(G)$ is also finitely generated, by $\operatorname{gr}_1(G) = G_{\mathrm{ab}}$. We let $\phi_k(G) = \operatorname{rank} \operatorname{gr}_k(G)$.
- Example: if F_n is the free group of rank n, then
  - $\operatorname{gr}(F_n)$ is the free Lie algebra $\operatorname{Lie}(\mathbb{Z}^n)$.
  - $\operatorname{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{s} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$.

MALCEV LIE ALGEBRAS

- The group-algebra $\mathbb{Q}G$ has a natural Hopf algebra structure, with comultiplication $\Delta(g) = g \otimes g$ and counit $\varepsilon\colon \mathbb{Q}G \to \mathbb{Q}$.
- (Quillen 1968) Let $I = \ker \varepsilon$. The I-adic completion $\widehat{\mathbb{Q}G} = \varprojlim_k \mathbb{Q}G/I^k$ is a filtered, complete Hopf algebra.
- An element $x \in \widehat{\Bbbk G}$ is called *primitive* if $\hat{\Delta} x = x \hat{\otimes} 1 + 1 \hat{\otimes} x$. The set of all such elements,

$$\mathfrak{m}(G) = \operatorname{Prim}(\widehat{\mathbb{Q}G}),$$

  with bracket $[x, y] = xy - yx$, is a complete, filtered Lie algebra, called the *Malcev Lie algebra* of G.
- Moreover, if we set $\operatorname{gr}_{\mathbb{Q}}(G) = \operatorname{gr}(G) \otimes \mathbb{Q}$, then

$$\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}_{\mathbb{Q}}(G).$$

- (Sullivan 1977) A finitely genetared group G is 1-formal if and only if $\mathfrak{m}(G)$ is a quadratic Lie algebra.

HOLONOMY LIE ALGEBRAS

- Let G be a finitely generated group, with G_{ab} torsion-free.
- Set $A^i = H^i(G, \mathbb{Z})$ and $A_i = (A^i)^* = \mathrm{Hom}(A^i, \mathbb{Z})$.
- The cup-product map $A^1 \otimes A^1 \to A^2$ factors through a linear map $\mu\colon A^1 \wedge A^1 \to A^2$.
- Dualizing, and identifying $(A^1 \wedge A^1)^* \cong A_1 \wedge A_1$, we obtain a linear map, $\mu^*\colon A_2 \to A_1 \wedge A_1 = \mathrm{Lie}_2(A_1)$.

DEFINITION (CHEN 1973, MARKL–PAPADIMA 1992)

The *holonomy Lie algebra* of G is $\mathfrak{h}(G) = \mathrm{Lie}(A_1)/\langle \mathrm{im}\, \mu^* \rangle$.

- $\mathfrak{h}(G)$ inherits a natural grading from $\mathrm{Lie}(A_1)$.
- $\mathfrak{h}(G)$ is a quadratic Lie algebra.
- There is a canonical surjection $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$, which is an isomorphism precisely when $\mathrm{gr}(G)$ is quadratic.

CHEN LIE ALGEBRAS

- The *Chen Lie algebra* of a group G is $\operatorname{gr}(G/G'')$, the associated graded Lie algebra of its maximal metabelian quotient.
- Assuming G is finitely generated, write $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ for the Chen ranks.
- (Chen 1951) $\theta_k(F_n) = \binom{n+k-2}{k}(k-1)$, for all $k \geqslant 2$.
- The projection $G \twoheadrightarrow G/G''$ induces $\operatorname{gr}(G) \twoheadrightarrow \operatorname{gr}(G/G'')$, and so $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- The map $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ induces $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \operatorname{gr}(G/G'')$.

THEOREM (PAPADIMA–S. 2004)

If G is 1-formal, then $\mathfrak{h}_{\mathbb{Q}}(G)/\mathfrak{h}_{\mathbb{Q}}(G)'' \xrightarrow{\simeq} \operatorname{gr}_{\mathbb{Q}}(G/G'')$.

Further improvements can be found in [S.–He Wang, 2017].

LIE ALGEBRAS OF A RAAG

Let $G = G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma) \rangle$ be the right-angled Artin group associated to a finite simple graph Γ.

THEOREM (DUCHAMP–KROB 1992, PAPADIMA–S. 2006)

- $\operatorname{gr}(G) \cong \mathfrak{h}(G)$.
- *The graded pieces are torsion-free, with ranks given by* $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = P_\Gamma(-t)$, *where* $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma)t^k$ *is the clique polynomial of* Γ, *with* $f_k(\Gamma) = \#\{k\text{-cliques of } \Gamma\}$.

THEOREM (PS 2006)

- $\operatorname{gr}(G/G'') \cong \mathfrak{h}(G)/\mathfrak{h}(G)''$.
- *The graded pieces are torsion-free, with ranks given by* $\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma(t/(1-t))$, *where* $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma)t^j$ *is the "cut polynomial" of* Γ, *with* $c_j(\Gamma) = \sum_{W\subset V:\, |W|=j} \tilde{b}_0(\Gamma_W)$.

THE RESCALING FORMULA

Let X be a connected space, and let Y be a simply-connected space (all spaces $\simeq$ to finite-type CW-complexes)

DEFINITION (PAPADIMA–S. 2004)

We say Y is a k*-rescaling* of X (over a ring R) if:

$$H^*(Y, R) \cong H^*(X, R)[k] \quad \text{as graded rings}$$

that is, $H^i(Y, R) \cong H^j(X, R)$ if $i = (2k+1)j$ and vanishes otherwise, and all isomorphisms compatible with cup products.

Examples of rescalings (over $R = \mathbb{Z}$)

- $X = S^1$, $Y = S^{2k+1}$
- $X = \#_1^g S^1 \times S^1$, $Y = \#_1^g S^{2k+1} \times S^{2k+1}$
- $X = \mathbb{C}^\ell \setminus \bigcup_{i=1}^n H_i$, $Y = \mathbb{C}^{(k+1)\ell} \setminus \bigcup_{i=1}^n H_i^{\times(k+1)}$, where $\mathcal{A} = \{H_1, \dots, H_n\}$ is a hyperplane arrangement in $\mathbb{C}^\ell$ and $\mathcal{A}^{k+1} := \{H_1^{\times(k+1)}, \dots, H_n^{\times(k+1)}\}$ (the *redundant* subspace arr.)

- For a graded Lie algebra L, its k-rescaling is the graded Lie algebra $L[k]$ with $L[k]_{2kq} = L_q$ and $L[k]_p = 0$ otherwise, and with Lie bracket rescaled accordingly.
- The *homotopy Lie algebra* of a simply-connected space Y is the graded Lie algebra $\pi_*(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r\geqslant 1} \pi_r(\Omega Y) \otimes \mathbb{Q}$, with Lie bracket coming from the Whitehead product.

THEOREM (PS 2004)

Let Y be a k-rescaling of X, and suppose $H^*(X,\mathbb{Q})$ is a Koszul algebra. Then:

- $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \operatorname{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$.
- *Set* $\Phi_r := \operatorname{rank} \pi_r(\Omega Y) = \operatorname{rank} \pi_{r+1}(Y)$. *Then* $\Phi_r = 0$ *if* $2k \nmid r$, *and*
$$\prod_{i\geqslant 1} \left(1 - t^{(2k+1)i}\right)^{\Phi_{2ki}} = \operatorname{Poin}_X(-t^k).$$

Consequently, $\operatorname{Poin}_{\Omega Y}(t) = \operatorname{Poin}_X(-t^{2k})^{-1}$.

ALGEBRAIC MODELS FOR SPACES

- For any (path-connected) space X, Sullivan defined a commutative differential graded algebra over $\mathbb{Q}$, denoted $A_{\mathrm{PL}}(X)$, such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic* (*q-*)*model* for X over a field $\Bbbk$ of characteristic 0 is a $\Bbbk$-cgda (A, d) which is (q-) equivalent (i.e., connected by a zig-zag of (q-) quasi-isomorphisms) to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- A cdga A is *formal* (or just *q-formal*) if it is (q-) equivalent to $(H^\bullet(A), d = 0)$.
- A CDGA A is of *finite-type* (or *q-finite*) if it is connected (i.e., $A^0 = \Bbbk \cdot 1$) and each graded piece A^i (with $i \leqslant q$) is finite-dimensional.
- Examples of spaces having finite-type models include:
    - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
    - Quasi-projective manifolds, compact solvmanifolds, and Sasakian manifolds, etc.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex, and $G = \pi_1(X)$.
- The algebra $R = \mathbb{C}[G_{\text{ab}}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1(X)} \times \mathrm{Tors}(G_{\text{ab}})$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- The algebraic sets $\mathcal{V}^i_s(X)$ are homotopy-type invariants of X.
- $\mathcal{V}^1_s(G) := \mathcal{V}^1_s(X)$ depend only on G; in fact, $\mathcal{V}^1_s(G) = \mathcal{V}^1_s(G/G'')$.
- These varieties can be arbitrarily complicated. E.g., if $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ is a Laurent polynomial with $f(1) = 0$, there is a f.p. group G with $G_{\text{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}^1_1(G) = \{f = 0\}$.

THEOREM (…, ARAPURA 1999, …, BUDUR–WANG 2015)

If X is a quasi-projective manifold, the varieties $\mathcal{V}^i_s(X)$ are finite unions of torsion-translates of subtori of $\mathrm{Char}(X)$.

RESONANCE VARIETIES

- Let $A=(A^{\bullet},\mathrm{d})$ be a connected, finite-type cdga over $\mathbb{C}$.
- For each $a\in Z^1(A)\cong H^1(A)$, we get a cochain complex,

$$(A^{\bullet},\delta_a)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i(u)=a\cdot u+\mathrm{d}\,u$, for all $u\in A^i$.
- The *resonance varieties* of A are the affine varieties

$$\mathcal{R}_s^i(A)=\{a\in H^1(A)\mid \dim H^i(A^{\bullet},\delta_a)\geqslant s\}.$$

- For a space X, the resonance varieties $\mathcal{R}_s^i(X):=\mathcal{R}_s^i(H^{\bullet}(X,\mathbb{C}))$ are homogeneous subsets of $H^1(X,\mathbb{C})$.

THE TANGENT CONE THEOREM

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- (DPS 2010) For a Zariski closed subset $W \subset H^1(X,\mathbb{C}^*)$, define

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- The exponential tangent cone $\tau_1(W)$ is a finite union of rationally defined linear subspaces included in $TC_1(W)$.

THEOREM (LIBGOBER 2002)

$TC_1(\mathcal{V}^i_s(X)) \subseteq \mathcal{R}^i_s(X)$.

THEOREM (DIMCA–PAPADIMA–S. 2010, DIMCA–PAPADIMA 2014)

Let X be a formal space. Then:

- *The map $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ induces isomorphisms of analytic germs, $\mathcal{R}^i_s(X,\mathbb{C})_{(0)} \xrightarrow{\simeq} \mathcal{V}^i_s(X)_{(1)}$.*
- $\tau_1(\mathcal{V}^i_s(X)) = TC_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X)$.

SPACES WITH FINITE MODELS

THEOREM

Let X be a connected CW-complex with finite q-skeleton. Assume X admits a q-finite q-model A. Then, for all $i \leqslant q$:

- *(Dimca–Papadima 2014)* $\mathcal{V}^i_s(X)_{(1)} \cong \mathcal{R}^i_s(A)_{(0)}$.
- *(Măcinic–Papadima–Popescu–S. 2017)* $\mathrm{TC}_0(\mathcal{R}^i_s(A)) \subseteq \mathcal{R}^i_s(X)$.
- *(Budur–Wang 2017) All irreducible components of* $\mathcal{V}^i_s(X)$ *passing through the identity of* $H^1(X, \mathbb{C}^*)$ *are* algebraic subtori.

EXAMPLE

Let G be a f.p. group with $G_{\mathrm{ab}} = \mathbb{Z}^n$ and $\mathcal{V}^1(G) = \{t \in (\mathbb{C}^*)^n \mid \sum_{i=1}^n t_i = n\}$. Then G admits no 1-finite 1-model.

THEOREM (PAPADIMA–S. 2017)

Let X be a space which admits a q-finite q-model. If $\mathcal{M}_q(X)$ is the Sullivan q-minimal model of X, then $b_i(\mathcal{M}_q(X)) < \infty$, for all $i \leqslant q+1$.

EXAMPLE

- Consider the free metabelian group $G = \mathrm{F}_n / \mathrm{F}_n''$ with $n \geqslant 2$.
- We have $\mathcal{V}^1(G) = \mathcal{V}^1(\mathrm{F}_n) = (\mathbb{C}^*)^n$, and so G passes the Budur–Wang test.
- But $b_2(\mathcal{M}_1(G)) = \infty$, and so G admits no 1-finite 1-model.

FINITENESS OBSTRUCTIONS FOR GROUPS

THEOREM (PAPADIMA–S 2017)

Let G be a metabelian group of the form $G = \pi/\pi''$, where π is a f.g. group which has a free, non-cyclic quotient. Then:

- *G is not finitely presentable.*
- *G does not admit a 1-finite 1-model.*

THEOREM (PS 2017)

A finitely generated group G admits a 1-finite 1-model if and only if its Malcev Lie algebra $\mathfrak{m}(G)$ is the LCS completion of a finitely presented Lie algebra.

BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- (Bieri–Neumann–Strebel 1987) For a f.g. group G, let

$$\Sigma^1(G) = \{\chi \in S(G) \mid \mathcal{C}_\chi(G) \text{ is connected}\},$$

  where $S(G) = (\operatorname{Hom}(G,\mathbb{R})\backslash\{0\})/\mathbb{R}^+$ and $\mathcal{C}_\chi(G)$ is the induced subgraph of $\operatorname{Cay}(G)$ on vertex set $G_\chi = \{g \in G \mid \chi(g) \geqslant 0\}$.
- $\Sigma^1(G)$ is an open set, independent of generating set for G.
- (Bieri, Renz 1988)

$$\Sigma^k(G,\mathbb{Z}) = \{\chi \in S(G) \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}.$$

  In particular, $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ for which G/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$

  where $S(G,N) = \{\chi \in S(G) \mid \chi(N) = 0\}$. In particular:

$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \iff \{\pm\chi\} \subseteq \Sigma^1(G).$$

Bounding the Σ-invariants

- The Σ-invariants were extended to spaces by Farber, Geoghegan, and Schütz (2010), using Novikov homology.
- For a connected CW-complex X with let $G = \pi_1(X)$, define

$$\Sigma^k(X,\mathbb{Z}) := \{\chi \in S(G) \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall\, i \leqslant k\}.$$

- Set $\tau_1^{\mathbb{R}}(W) = \tau_1(W) \cap H^1(X,\mathbb{R})$ and $\mathcal{W}^i(X) = \bigcup_{q\leqslant i} \mathcal{V}_1^q(X)$.

Theorem (Papadima–S. 2010)

$$\Sigma^i(X,\mathbb{Z}) \subseteq S(G) \backslash S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))).$$

- If X is formal, we may replace $\tau_1^{\mathbb{R}}(\mathcal{W}^i(X))$ with $\bigcup_{q\leqslant i} \mathcal{R}_1^q(X,\mathbb{R})$.
- (PS 2006/09) Equality holds for RAAGs and toric complexes.
- (Koban–McCammond–Meier 2015) Equality holds for the pure braid groups P_n in degree $i = 1$.

KOLLÁR'S QUESTION

Two groups, G_1 and G_2, are said to be *commensurable up to finite kernels* if there is a zig-zag of homomorphisms,

$$\begin{array}{ccccccccc} G_1 & & H_2 & & & & \cdots & & G_2\,, \\ & \nwarrow\; H_1 \;\nearrow & & \nwarrow & \cdots & & \nwarrow\; H_q \;\nearrow & & \end{array}$$

with all arrows of finite kernel and cofinite image.

QUESTION (J. KOLLÁR 1995)

Given a smooth, projective variety M, is the group $G = \pi_1(M)$ commensurable, up to finite kernels, with another group, π, admitting a $K(\pi, 1)$ which is a quasi-projective variety?

THEOREM (DIMCA–PAPADIMA–S. 2009)

For each $k \geqslant 3$, there is a smooth, irreducible, complex projective variety M of complex dimension $k-1$, such that $\pi_1(M)$ is of type F_{k-1}, but not of type FP_k.

HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension 1 linear subspaces in a finite-dimensional $\mathbb{C}$-vector space V.
- The *intersection lattice*, $L(\mathcal{A})$, is the poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- The *complement*, $M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$, is a connected, smooth quasi-projective variety, and also a Stein manifold.
- The fundamental group $\pi = \pi_1(M(\mathcal{A}))$ admits a finite presentation, with generators x_H for each $H \in \mathcal{A}$.
- Set $U(\mathcal{A}) = \mathbb{P}(M(\mathcal{A}))$. Then $M(\mathcal{A}) \cong U(\mathcal{A}) \times \mathbb{C}^*$.

THEOREM (DIMCA–PAPADIMA 2003)

$M(\mathcal{A})$ *has the homotopy type of a* minimal *CW-complex.*

This solved a conjecture made by Papadima–S. at MFO in 1999.

COHOMOLOGY RING

- The logarithmic 1-form $\omega_H = \frac{1}{2\pi \mathrm{i}} \,\mathrm{d}\log \alpha_H \in \Omega_{\mathrm{dR}}(M)$ is a closed form, representing a class $e_H \in H^1(M,\mathbb{Z})$.
- Let E be the $\mathbb{Z}$-exterior algebra on $\{e_H \mid H \in \mathcal{A}\}$, and let $\partial\colon E^\bullet \to E^{\bullet-1}$ be the differential given by $\partial(e_H) = 1$.
- The ring $H^\bullet(M(\mathcal{A}),\mathbb{Z})$ is isomorphic to the OS-algebra E/I, where

$$I = \text{ideal}\left\{\partial\Big(\prod_{H\in\mathcal{B}} e_H\Big) \,\Big|\, \mathcal{B} \subseteq \mathcal{A} \text{ and } \operatorname{codim} \bigcap_{H\in\mathcal{B}} H < |\mathcal{B}| \right\}.$$

- Hence, the map $e_H \mapsto \omega_H$ extends to a cdga quasi-isomorphism, $\omega\colon (H^\bullet(M,\mathbb{R}), \mathrm{d}=0) \longrightarrow \Omega^\bullet_{\mathrm{dR}}(M)$.
- Therefore, $M(\mathcal{A})$ is formal.
- $M(\mathcal{A})$ is minimally pure (i.e., $H^k(M(\mathcal{A}),\mathbb{Q})$ is pure of weight $2k$, for all k), which again implies formality (Dupont 2016).

MULTINETS AND DEGREE 1 RESONANCE

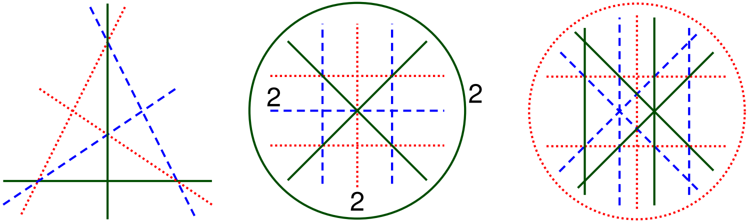


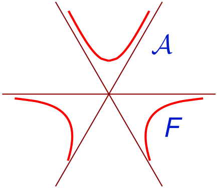
FIGURE: $(3,2)$-net; $(3,4)$-multinet; non-3-net, reduced $(3,4)$-multinet

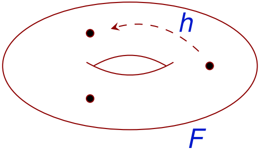
THEOREM (FALK, COHEN–S., LIBGOBER–YUZVINSKY, FALK–YUZ)

$$\mathcal{R}^1_s(M(\mathcal{A}),\mathbb{C}) = \bigcup_{\mathcal{B}\subseteq\mathcal{A}} \bigcup_{\substack{\mathcal{N} \text{ a } k\text{-multinet on } \mathcal{B} \\ \text{with at least } s+2 \text{ parts}}} P_{\mathcal{N}}.$$

where $P_{\mathcal{N}}$ is the $(k-1)$-dimensional linear subspace spanned by the vectors $u_2 - u_1, \dots, u_k - u_1$, where $u_\alpha = \sum_{H\in\mathcal{B}_\alpha} m_H e_H$.

MILNOR FIBRATION





- Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^{d+1}$. For each $H \in \mathcal{A}$ let α_H be a linear form with $\ker(\alpha_H) = H$, and let $Q = \prod_{H \in \mathcal{A}} \alpha_H$.
- $Q\colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a smooth fibration, $Q\colon M(\mathcal{A}) \to \mathbb{C}^*$. The *Milnor fiber* of the arrangement is $F(\mathcal{A}) := Q^{-1}(1)$.
- F is a Stein manifold. It has the homotopy type of a finite cell complex of dim d. In general, F is neither formal, nor minimal.
- $F = F(\mathcal{A})$ is the regular, $\mathbb{Z}_n$-cover of $U = U(\mathcal{A})$, classified by the morphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ taking each loop x_H to 1.

Modular inequalities

- The monodromy diffeo, $h\colon F \to F$, is given by $h(z) = e^{2\pi \mathrm{i}/n} z$.
- Let $\Delta(t)$ be the characteristic polynomial of $h_*\colon H_1(F, \mathbb{C}) \circlearrowleft$. Since $h^n = \mathrm{id}$, we have

$$\Delta(t) = \prod_{r|n} \Phi_r(t)^{e_r(\mathcal{A})},$$

  where $\Phi_r(t)$ is the r-th cyclotomic polynomial, and $e_r(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.
- WLOG, we may assume $d = 2$, so that $\bar{\mathcal{A}} = \mathbb{P}(\mathcal{A})$ is an arrangement of lines in $\mathbb{CP}^2$.
- If there is no point of $\bar{\mathcal{A}}$ of multiplicity $q \geqslant 3$ such that $r \mid q$, then $e_r(\mathcal{A}) = 0$ (Libgober 2002).
- In particular, if $\bar{\mathcal{A}}$ has only points of multiplicity 2 and 3, then $\Delta(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$. If multiplicity 4 appears, then we also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

- Let $A = H^{\bullet}(M(\mathcal{A}), \Bbbk)$, and let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$.
- Assume $\Bbbk$ has characteristic $p > 0$, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^m}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $m \geqslant 1$.

THEOREM (PAPADIMA–S. 2017)

- *Suppose $\mathcal{A}$ admits a k-net. Then $\beta_p(\mathcal{A}) = 0$ if $p \nmid k$ and $\beta_p(\mathcal{A}) \geqslant k-2$, otherwise.*
- *If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k-2$.*

COMBINATORICS AND MONODROMY

THEOREM (PAPADIMA–S. 2017)

Suppose $\bar{\mathcal{A}}$ has no points of multiplicity $3r$ with $r > 1$. TFAE:

- *$\mathcal{A}$ admits a reduced 3-multinet.*
- *$\mathcal{A}$ admits a 3-net.*
- *$\beta_3(\mathcal{A}) \neq 0$.*

Moreover, the following hold:

- *$\beta_3(\mathcal{A}) \leqslant 2$.*
- *$e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$.*

In particular, if $\bar{\mathcal{A}}$ has only double and triple points, then $\Delta(t)$ is combinatorially determined.

THEOREM (PS 2017)

Suppose $\mathcal{A}$ supports a 4-net and $\beta_2(\mathcal{A}) \leqslant 2$. Then

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) = 2.$$

CONJECTURE (PAPADIMA–S. 2017)

The characteristic polynomial of the degree 1 algebraic monodromy for the Milnor fibration of an arrangement $\mathcal{A}$ of rank at least 3 is given by the combinatorial formula

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements, such as:

- All sub-arrangements of non-exceptional Coxeter arrangements (Măcinic, Papadima).
- All complex reflection arrangements (Măcinic, Papadima, Popescu, Dimca, Sticlaru).
- Certain types of complexified real arrangements (Yoshinaga, Bailet, Torielli, Settepanella).

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