MILNOR FIBRATIONS OF HYPERPLANE ARRANGEMENTS

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HYPERPLANE ARRANGEMENTS

- An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in $\mathbb{C}^\ell$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.
- *Complement*: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0,1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1,\dots,n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on n strings).

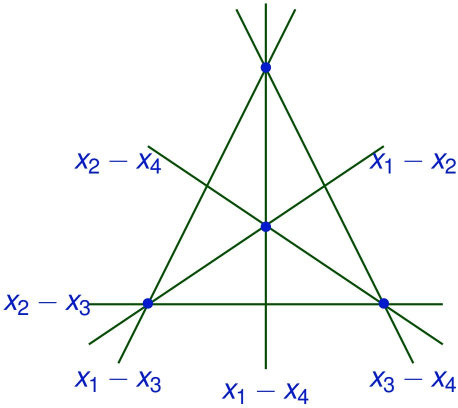


FIGURE : A planar slice of the braid arrangement $\mathcal{A}_4$

- We may assume that $\mathcal{A}$ is essential, i.e., $\bigcap_{H\in\mathcal{A}} H = \{0\}$.
- Fix an ordering $\mathcal{A} = \{H_1, \dots, H_n\}$, and choose linear forms $f_i \colon \mathbb{C}^\ell \to \mathbb{C}$ with $\ker(f_i) = H_i$.
- Define an injective linear map

$$\iota_{\mathcal{A}} \colon \mathbb{C}^\ell \to \mathbb{C}^n, \quad z \mapsto (f_1(z), \dots, f_n(z)).$$

- This map restricts to an inclusion $\iota\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{B}_n)$. Thus,

$$M(\mathcal{A}) = \iota_{\mathcal{A}}(\mathbb{C}^\ell) \cap (\mathbb{C}^*)^n,$$

  a “very affine" subvariety of $(\mathbb{C}^*)^n$.
- The tropicalization of this sub variety is a fan in $\mathbb{R}^n$. Feichtner and Sturmfels: this is the Bergman fan of $L(\mathcal{A})$.

- $M(\mathcal{A})$ has the homotopy type of a connected, finite cell complex of dimension ℓ.
- In fact, $M = M(\mathcal{A})$ admits a *minimal* cell structure. Consequently, $H_*(M, \mathbb{Z})$ is torsion-free.
- The Betti numbers $b_q(M) := \operatorname{rank} H_q(M, \mathbb{Z})$ are given by

$$\sum_{q=0}^{\ell} b_q(M) t^q = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)},$$

  where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function, defined recursively by $\mu(\mathbb{C}^\ell) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$.
- The Orlik–Solomon algebra $H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra on generators $\{e_H \mid H \in \mathcal{A}\}$ by an ideal determined by the circuits in the matroid of $\mathcal{A}$.
- Thus, the ring $H^*(M, \Bbbk)$ is determined by $L(\mathcal{A})$, for every field $\Bbbk$.

THE MILNOR FIBRATION(S) OF AN ARRANGEMENT

- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^\ell \to \mathbb{C}$ be a linear form with kernel H.
- For each choice of multiplicities $m = (m_H)_{H\in\mathcal{A}}$ with $m_H \in \mathbb{N}$, let

$$Q_m := Q_m(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H^{m_H},$$

  a homogeneous polynomial of degree $N = \sum_{H\in\mathcal{A}} m_H$.
- The map $Q_m \colon \mathbb{C}^\ell \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the *Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{Q_m} \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.
- $F_m(\mathcal{A})$ has the homotopy type of a finite cell complex, with $\gcd(m)$ connected components, and of dimension $\ell - 1$.
- The *(geometric) monodromy* is the diffeomorphism

$$h\colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi \mathrm{i}/N} z.$$

- If all $m_H = 1$, the polynomial $Q = Q_m(\mathcal{A})$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

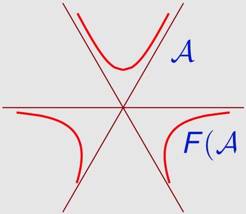
EXAMPLE

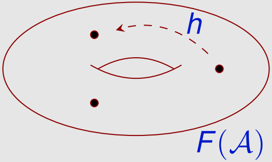
Let $\mathcal{A}$ be the single hyperplane $\{0\}$ inside $\mathbb{C}$. Then:

- $M(\mathcal{A}) = \mathbb{C}^*$.
- $Q_m(\mathcal{A}) = z^m$.
- $F_m(\mathcal{A}) = m$-roots of 1.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:





More generally, if $\mathcal{A}$ is a pencil of n lines in $\mathbb{C}^2$, then $F(\mathcal{A})$ is a Riemann surface of genus $\binom{n-1}{2}$, with n punctures.

- Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q_m(\mathcal{B}_n) = z_1^{m_1} \cdots z_n^{m_n}$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and

$$F_m(\mathcal{B}_n) = \ker(Q_m) \cong (\mathbb{C}^*)^{n-1} \times \mathbb{Z}_{\gcd(m)}$$

- Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an essential arrangement. The inclusion $\iota_{\mathcal{A}} \colon M(\mathcal{A}) \to M(\mathcal{B}_n)$ restricts to a bundle map

$$\begin{array}{ccccc}
F_m(\mathcal{A}) & \longrightarrow & M(\mathcal{A}) & \xrightarrow{Q_m(\mathcal{A})} & \mathbb{C}^* \\
\downarrow & & \downarrow{\scriptstyle \iota_{\mathcal{A}}} & & \| \\
F_m(\mathcal{B}_n) & \longrightarrow & M(\mathcal{B}_n) & \xrightarrow{Q_m(\mathcal{B}_n)} & \mathbb{C}^*
\end{array}$$

- Thus,

$$F_m(\mathcal{A}) = M(\mathcal{A}) \cap F_m(\mathcal{B}_n)$$

- The tropicalization of $F_m(\mathcal{A})$ is a fan in $\mathbb{R}^{n-1}$. Question: Is this fan determined by $L(\mathcal{A})$ (and the multiplicity vector m)?

THE HOMOLOGY OF THE MILNOR FIBER

Two basic questions about the topology of the Milnor fibration(s):

(Q1) Are the homology groups $H_q(F(\mathcal{A}), \mathbb{C})$ determined by $L(\mathcal{A})$?
If so, is the characteristic polynomial of the algebraic monodromy, $h_*\colon H_q(F(\mathcal{A}), \mathbb{C}) \to H_q(F(\mathcal{A}), \mathbb{C})$, also determined by $L(\mathcal{A})$?

(Q2) Are the homology groups $H_q(F(\mathcal{A}), \mathbb{Z})$ torsion-free?
If so, does $F(\mathcal{A})$ admit a minimal cell structure?

In this talk, I will indicate some recent progress on these questions:

- A partial, positive answer to (Q1): joint work with Stefan Papadima (in progress).
- A negative answer to (Q2): joint work with Graham Denham (to appear in PLMS).

ALGEBRAIC MONODROMY

- Recall: the monodromy $h\colon F(\mathcal{A}) \to F(\mathcal{A})$ has order $n = |\mathcal{A}|$.
- Thus, the characteristic polynomial of h_* acting on $H_1(F(\mathcal{A}), \mathbb{C})$ is

$$\Delta(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$

where $\Phi_1 = t - 1$, $\Phi_2 = t + 1$, $\Phi_3 = t^2 + t + 1, \dots$
are the cyclotomic polynomials, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

- Easy to see: $e_1(\mathcal{A}) = n - 1$. Thus, for $q = 1$, question (Q1) is equivalent to: are the integers $e_d(\mathcal{A})$ determined by $L_{\leqslant 2}(\mathcal{A})$?

THEOREM (PAPADIMA–S. 2013)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, for some $r > 1$. Then $e_3(\mathcal{A}) \leqslant 2$ and $e_3(\mathcal{A})$ is combinatorially determined.

A similar result holds for $e_2(\mathcal{A})$.

- Let $A^* = H^*(M(\mathcal{A}), \Bbbk)$, where $\Bbbk$ is a field of characteristic $p > 0$.
- Let $\sigma = \sum_{H\in\mathcal{A}} e_H \in A^1$ be the "diagonal" vector, and consider the $\Bbbk$-vector space $Z(\sigma) = \{\tau \in A^1 \mid \sigma \cup \tau = 0 \in A^2\}$.
- Define the mod-p Aomoto-Betti number of $\mathcal{A}$ as

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} Z(\sigma) - 1.$$

- $\beta_p(\mathcal{A})$ depends only on $L(\mathcal{A})$ and p, and $0 \leqslant \beta_p(\mathcal{A}) \leqslant |\mathcal{A}| - 2$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $s \geqslant 1$.

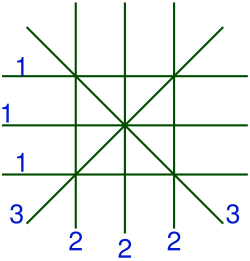
THEOREM (PAPADIMA–S. 2013)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, for some $r > 1$. Then $\beta_3(\mathcal{A}) \leqslant 2$ and $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$.

TORSION IN HOMOLOGY

THEOREM (COHEN–DENHAM–S. 2003)

For every prime $p \geqslant 2$, there is a multi-arrangement $(\mathcal{A}, m)$ such that $H_1(F_m(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion.



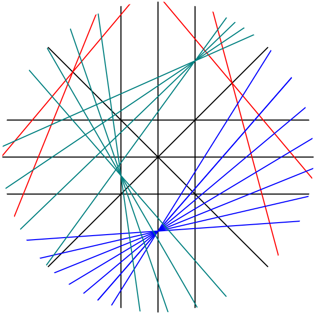
Simplest example: the arrangement of 8 hyperplanes in $\mathbb{C}^3$ with

$$Q_m(\mathcal{A}) = x^2y(x^2-y^2)^3(x^2-z^2)^2(y^2-z^2)$$

Then $H_1(F_m(\mathcal{A}), \mathbb{Z}) = \mathbb{Z}^7 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

THEOREM (DENHAM–S. 2013)

For every prime $p \geqslant 2$, there is an arrangement $\mathcal{A}$ such that $H_q(F(\mathcal{A}), \mathbb{Z})$ has non-zero p-torsion, for some $q > 1$.



Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2-y^2)(x^2-z^2)(y^2-z^2)w_1w_2w_3w_4w_5(x^2-w_1^2)(x^2-2w_1^2)(x^2-3w_1^2)(x-4w_1)\cdot$$

$$((x-y)^2-w_2^2)((x+y)^2-w_3^2)((x-z)^2-w_4^2)((x-z)^2-2w_4^2)\cdot((x+z)^2-w_5^2)((x+z)^2-2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

MULTINETS

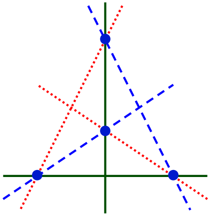
- Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.
- $L_1(\mathcal{A}) \longleftrightarrow$ lines of $\bar{\mathcal{A}}$, $L_2(\mathcal{A}) \longleftrightarrow$ intersection points of $\bar{\mathcal{A}}$, poset structure of $L_{\leqslant 2}(\mathcal{A}) \longleftrightarrow$ incidence structure of $\bar{\mathcal{A}}$.
- A flat $X \in L_2(\mathcal{A})$ has multiplicity q if the point $\bar{X}$ has exactly q lines from $\bar{\mathcal{A}}$ passing through it.

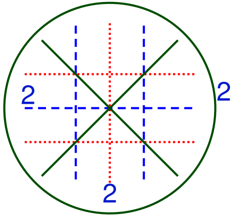
DEFINITION (FALK AND YUZVINSKY)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m\colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

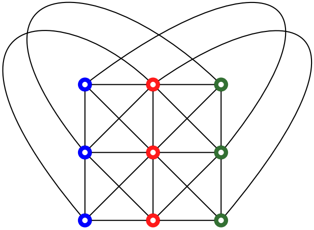
1. There is an integer d such that $\sum_{H\in\mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H\in\mathcal{A}_\alpha : H\supset X} m_H$ is independent of α.
4. Each $\left(\bigcup_{H\in\mathcal{A}_\alpha} H\right)\backslash\mathcal{X}$ is connected.

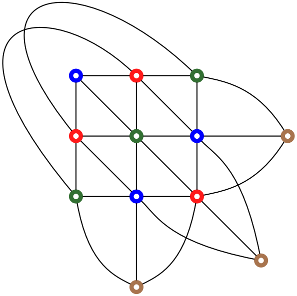
- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- If $m_H = 1$, for all $H \in \mathcal{A}$, the multinet is *reduced*.
- If, furthermore, $n_X = 1$, for all $X \in \mathcal{X}$, this is a *net*. In this case, $|\mathcal{A}_\alpha| = |\mathcal{A}|\,/\,k = d$, for all α, and $\left|\bar{\mathcal{X}}\right| = d^2$.
- If $\mathcal{A}$ admits a $(3, d)$-net, then the d^2 multi-colored triple points define a Latin square.
- More generally, the base locus of a (k, d)-net is encoded by a $(k-2)$-tuple of orthogonal Latin squares.





A $(3,2)$-net on the A_3 arrangement. A $(3,4)$-multinet on the B_3 arrangement.





A $(3,3)$-net on the Ceva matroid. A $(4,3)$-net on the Hessian matroid.

- If $\mathcal{A}$ has no flats of multiplicity kr, for some $r > 1$, then every reduced k-multinet is a k-net.
- (Yuzvinsky and Pereira–Yuzvinsky): If $\mathcal{A}$ supports a k-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; moreover, if the multinet is not reduced, then $k = 3$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4, 3)$-net.

MULTINETS AND $H_1(F(\mathcal{A}),\mathbb{C})$

Recall that

$$H_1(F(\mathcal{A}),\mathbb{C}) = \mathbb{C}^{n-1} \oplus \bigoplus_{1<d|n} (\mathbb{C}[t]/\Phi_d(t))^{e_d(\mathcal{A})}.$$

(as a module over $\mathbb{C}[\mathbb{Z}_n]$, where $n = |\mathcal{A}|$), and so

$$b_1(F(\mathcal{A})) = n - 1 + \sum_{1<d|n} \varphi(d) e_d(\mathcal{A}).$$

Using the theory of cohomology jumping loci, which is well understood for complements of hyperplane arrangements (at least in degree 1), one can prove the following.

PROPOSITION

If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k - 2$.

LEMMA (PS)

If $\mathcal{A}$ supports a 3-net with parts $\mathcal{A}_\alpha$, then:

1. $1 \leqslant \beta_3(\mathcal{A}) \leqslant \beta_3(\mathcal{A}_\alpha) + 1$, *for all* α.
2. *If* $\beta_3(\mathcal{A}_\alpha) = 0$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$.
3. *If* $\beta_3(\mathcal{A}_\alpha) = 1$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$ *or* 2.

All possibilities do occur:

- Braid arrangement: has a $(3, 2)$-net from the Latin square of $\mathbb{Z}_2$. $\beta_3(\mathcal{A}_\alpha) = 0$ $(\forall\alpha)$ and $\beta_3(\mathcal{A}) = 1$.
- Pappus arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_1) = \beta_3(\mathcal{A}_2) = 0$, $\beta_3(\mathcal{A}_3) = 1$ and $\beta_3(\mathcal{A}) = 1$.
- Ceva arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_\alpha) = 1$ $(\forall\alpha)$ and $\beta_3(\mathcal{A}) = 2$.

THEOREM (PS)

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, for some $r > 1$. Then $\beta_3(\mathcal{A}) \leqslant 2$. Moreover, the following conditions are equivalent:

1. *$\mathcal{A}$ admits a reduced 3-multinet.*
2. *$\mathcal{A}$ admits a 3-net.*
3. *$\beta_3(\mathcal{A}) \neq 0$.*

REMARK

- One may define $\beta_p(\mathcal{M})$ for any matroid $\mathcal{M}$.
- For each $n \in \mathbb{N}$, there exists a matroid $\mathcal{M}_n$ supporting a $(3, 3^n)$-net corresponding to $\mathbb{Z}_3^n$, such that $\beta_3(\mathcal{M}_n) = n + 1$.
- By the above, such a matroid is realizable by an arrangement in $\mathbb{C}^3$ if and only if $n = 1$.

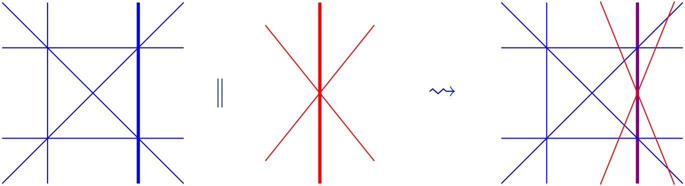
Multinets and torsion in $H_*(F(\mathcal{A}),\mathbb{Z})$

A *pointed multinet* on an arrangement $\mathcal{A}$ is a multinet structure, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

Theorem (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_1(F_{m'}(\mathcal{A}'),\mathbb{Z})$ has non-zero p-torsion.

- This torsion is explained by the fact that the geometry of the cohomology jumping loci of $M(\mathcal{A}')$ varies with the characteristic of the ground field.
- To produce p-torsion in the homology of the usual Milnor fiber, we use a “polarization" construction, based on the “parallel connection" construction of Falk and Proudfoot (2002).



$(\mathcal{A}, m) \rightsquigarrow \mathcal{A} \| m$, an arrangement of $N = \sum_{H\in\mathcal{A}} m_H$ hyperplanes, of rank equal to $\operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} : m_H \geqslant 2\}|$.

THEOREM (DS)

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H.

There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A}\backslash\{H\}$ such that $H_q(F(\mathcal{B}),\mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}'\|m'$ and $q = 1 + \left|\left\{K \in \mathcal{A}' : m'_K \geqslant 3\right\}\right|$.

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