Duality and resonance

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POINCARÉ DUALITY ALGEBRAS

- Let A be a graded, graded-commutative algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i\geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - $\cdot\colon A^i \otimes A^j \to A^{i+j}$.
  - $ab = (-1)^{ij} ba$ for all $a \in A^i$, $b \in B^j$.
- We will assume that A is connected ($A^0 = \Bbbk \cdot 1$), and locally finite (all the Betti numbers $b_i(A) := \dim_{\Bbbk} A^i$ are finite).
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk,\ a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- Consequently,
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*, \mathrm{PD}(a)(b) = \varepsilon(ab)$ are isomorphisms.
  - Each $a \in A^i$ has a *Poincaré dual*, $a^{\vee} \in A^{m-i}$, such that $\varepsilon(aa^{\vee}) = 1$.
  - The *orientation class* is defined as $\omega_A = 1^{\vee}$, so that $\varepsilon(\omega_A) = 1$.

The associated alternating form

- Associated to a $\Bbbk$-PD_m algebra there is an alternating m-form,
$$\mu_A\colon \bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$
- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the PD basis for A^2.
- The multiplication in A, then, is given on basis elements by
$$e_i e_j = \sum_{k=1}^n \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$
where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.
- Alternatively, let $A_i = (A^i)^*$, and let $e^i \in A_1$ be the (Kronecker) dual of e_i. We may then view μ dually as a trivector,
$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$
which encodes the algebra structure of A.

POINCARÉ DUALITY IN ORIENTABLE MANIFOLDS

- If M is a compact, connected, orientable, m-dimensional manifold, then the cohomology ring $A = H^{\bullet}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via “Borromean surgery."

- If M bounds an oriented 4-manifold W such that the cup-product pairing on $H^2(W, M)$ is non-degenerate (e.g., if M is the link of an isolated surface singularity), then $\mu_M = 0$.

RESONANCE VARIETIES OF GRADED ALGEBRAS

- Let A be a connected, finite-type cga over $\Bbbk = \mathbb{C}$.
- For each $a \in A^1$, there is a cochain complex of $\Bbbk$-vector spaces,

$$(A, \delta_a)\colon \; A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a(b) = a \cdot b$, for $b \in A^i$.
- The *resonance varieties* of A are the sets

$$\mathcal{R}_s^i(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, \delta_a) \geqslant s\}.$$

- An element $a \in A^1$ belongs to $\mathcal{R}_s^i(A)$ if and only if

$$\operatorname{rank} \delta_a^{i+1} + \operatorname{rank} \delta_a^i \leqslant b_i(A) - s.$$

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^*$.
- Identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.
- Define a cochain complex of free S-modules, $\mathbf{L}(A) := (A^\bullet \otimes S, \delta)$,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta^i} A^{i+1} \otimes S \xrightarrow{\delta^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots,$$

  where $\delta^i(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.
- The specialization of $(A \otimes S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).
- Hence, $\mathcal{R}^i_s(A)$ is the zero-set of the ideal generated by all minors of size $b_i - s + 1$ of the block-matrix $\delta^{i+1} \oplus \delta^i$.
- In particular, $\mathcal{R}^1_s(A) = V(I_{n-s}(\delta^1))$, the zero-set of the ideal of codimension s minors of δ^1.

EXAMPLE (EXTERIOR ALGEBRA)

Let $E = \bigwedge V$, where $V = \Bbbk^n$, and $S = \mathrm{Sym}(V)$. Then $\mathbf{L}(E)$ is the Koszul complex on V. E.g., for $n = 3$:

$$S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}} S\,.$$

This chain complex provides a free resolution $\varepsilon\colon \mathbf{L}(E) \to \Bbbk$ of the trivial S-module $\Bbbk$. Hence,

$$\mathcal{R}^i_s(E) = \begin{cases} \{0\} & \text{if } s \leqslant \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example (Non-zero resonance)

Let $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2 \rangle$, and set $S = \Bbbk[x_1, x_2, x_3]$. Then

$$\mathbf{L}(A): \quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 & 0 \\ 0 & x_3 \\ -x_1 & -x_2 \end{pmatrix}} S^2 .$$

$$\mathcal{R}^1_s(A) = \begin{cases} \{x_3 = 0\} & \text{if } s = 1, \\ \{0\} & \text{if } s = 2 \text{ or } 3, \\ \emptyset & \text{if } s > 3. \end{cases}$$

Example (Non-linear resonance)

Let $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle$. Then

$$\mathbf{L}(A): \quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_4 & 0 & -x_2 \\ 0 & x_3 & x_1 \\ 0 & -x_2 & x_4 \\ -x_1 & 0 & -x_3 \end{pmatrix}} S^3 .$$

$$\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$$

Properties of resonance

- Product formula

$$\mathcal{R}^i_s(B\otimes C) = \begin{cases} \mathcal{R}^1_s(B)\times\{0\}\cup\{0\}\times\mathcal{R}^1_s(C), & \text{if } i=1,\\ \bigcup_{k+\ell=i}\mathcal{R}^k_1(B)\times\mathcal{R}^\ell_1(C), & \text{if } i\geqslant 2 \text{ and } s=1. \end{cases}$$

- Coproduct formula

$$\mathcal{R}^i_s(B\vee C) = \begin{cases} \bigcup_{k+\ell=s-1}(\mathcal{R}^1_k(B)\backslash\{0\})\times(\mathcal{R}^1_\ell(C)\backslash\{0\})\cup \\ \quad (\{0\}\times\mathcal{R}^1_{s-\dim B^1}(C))\cup(\mathcal{R}^1_{s-\dim C^1}(B)\times\{0\}), & \text{if } i=1,\\ \bigcup_{k+\ell=s}\mathcal{R}^i_k(B)\times\mathcal{R}^i_\ell(C), & \text{if } i\geqslant 2. \end{cases}$$

- If $\varphi\colon A\to B$ is a cga morphism such that $\varphi_1\colon A^1\to B^1$ is injective, then $\varphi_1(\mathcal{R}^1_s(A))\subseteq\mathcal{R}^1_s(B)$, for all $s\geqslant 0$.
- In general, $\varphi_1(\mathcal{R}^i_s(A))\nsubseteq\mathcal{R}^i_s(B)$, even if φ is injective.

Resonance varieties of PD-algebras

- Let A be a PD_m algebra.
- For all $0 \leqslant i \leqslant m$ and all $a \in A^1$, the square

$$\begin{array}{ccc} (A^{m-i})^* & \xrightarrow{(\delta_a^{m-i-1})^*} & (A^{m-i-1})^* \\ \mathrm{PD}\uparrow\cong & & \mathrm{PD}\uparrow\cong \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1} \end{array}$$

  commutes up to a sign of $(-1)^i$.
- Consequently,

$$\left(H^i(A, \delta_a)\right)^* \cong H^{m-i}(A, \delta_{-a}).$$

- Hence, for all i and s,

$$\mathcal{R}^i_s(A) = \mathcal{R}^{m-i}_s(A).$$

- In particular, $\mathcal{R}^m_1(A) = \{0\}$.

DEGREE 1 MAPS

- Let A and B be two PD_m algebras. A morphism $\varphi\colon A \to B$ of cga's has *degree* 1 if the linear map $\varphi_m\colon A^m \to B^m$ is non-zero.
- We may then pick orientation classes such that $\varphi_m(\omega_A) = \omega_B$.

PROPOSITION

Let $\varphi\colon A \to B$ be a degree 1 map between two PD_m algebras. Then:

- $\varphi(a^\vee) = \varphi(a)^\vee$, *for all homogeneous elements $a \in A$.*
- *The map φ is injective.*
- *For all $a \in A^1$, the map φ induces a homomorphism*

$$\varphi^*\colon H^*(A, \delta_a) \to H^*(B, \delta_{\varphi_1(a)}).$$

- *The map $\varphi_1\colon A^1 \hookrightarrow B^1$ restricts to inclusions $\mathcal{R}^i_s(A) \hookrightarrow \mathcal{R}^i_s(B)$.*

3-DIMENSIONAL POINCARÉ DUALITY ALGEBRAS

- Let A be a PD_3-algebra with $b_1(A) = n > 0$. Then
  - $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$.
  - $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ for $1 \leqslant s \leqslant n$.
  - $\mathcal{R}^i_s(A) = \emptyset$, otherwise.
- Write $\mathcal{R}_s(A) = \mathcal{R}^1_s(A)$. Work of Buchsbaum and Eisenbud on Pfaffians of skew-symmetric matrices implies that
  - $\mathcal{R}_{2k}(A) = \mathcal{R}_{2k+1}(A)$ if n is even.
  - $\mathcal{R}_{2k-1}(A) = \mathcal{R}_{2k}(A)$ if n is odd.
- If μ_A has rank $n \geqslant 3$, then $\mathcal{R}_{n-2}(A) = \mathcal{R}_{n-1}(A) = \mathcal{R}_n(A) = \{0\}$.
  - Here, the *rank* of a form $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$.
  - The *nullity* of μ is the maximum dimension of a subspace $U \subset V$ such that $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.

- If $n \geqslant 4$, then $\dim \mathcal{R}_1(A) \geqslant \operatorname{null}(\mu_A) \geqslant 2$.
- If n is even, then $\mathcal{R}_1(A) = \mathcal{R}_0(A) = A^1$.
- If $n = 2g+1 > 1$, then $\mathcal{R}_1(A) \neq A^1$ if and only if μ_A is 'generic' in the sense of Berceanu and Papadima (1994).
- That is, $\exists\ c \in A^1$ such that the 2-form $\gamma_c \in \bigwedge^2 A_1$ given by $\gamma_c(a \wedge b) = \mu_A(a \wedge b \wedge c)$ has rank $2g$, i.e., $\gamma_c^g \neq 0$ in $\bigwedge^{2g} A_1$.
- In that case, $\mathcal{R}_1(A)$ is the hypersurface $\operatorname{Pf}(\mu_A) = 0$, where $\operatorname{pf}(\delta^1(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$.

EXAMPLE

Let $M = S^1 \times \Sigma_g$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^{g} a_i b_i c$ is generic, and $\operatorname{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}_1 = \cdots = \mathcal{R}_{2g-2} = \{x_{2g+1} = 0\}$ and $\mathcal{R}_{2g-1} = \mathcal{R}_{2g} = \mathcal{R}_{2g+1} = \{0\}$.

RESONANCE VARIETIES OF 3-FORMS OF LOW RANK

n	μ	$\mathcal{R}_1$
3	123	0

n	μ	$\mathcal{R}_1=\mathcal{R}_2$	$\mathcal{R}_3$
5	125+345$^{\otimes}$	$\{x_5=0\}$	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2=\mathcal{R}_3$	$\mathcal{R}_4$
6	123+456$^{\#}$	$\mathbb{C}^6$	$\{x_1=x_2=x_3=0\}\cup\{x_4=x_5=x_6=0\}$	0
	123+236+456	$\mathbb{C}^6$	$\{x_3=x_5=x_6=0\}$	0

n	μ	$\mathcal{R}_1=\mathcal{R}_2$	$\mathcal{R}_3=\mathcal{R}_4$	$\mathcal{R}_5$
7	147+257+367$^{\otimes}$	$\{x_7=0\}$	$\{x_7=0\}$	0
	456+147+257+367	$\{x_7=0\}$	$\{x_4=x_5=x_6=x_7=0\}$	0
	123+456+147	$\{x_1=0\}\cup\{x_4=0\}$	$\{x_1=x_2=x_3=x_4=0\}\cup\{x_1=x_4=x_5=x_6=0\}$	0
	123+456+147+257	$\{x_1x_4+x_2x_5=0\}$	$\{x_1=x_2=x_4=x_5=x_7^2-x_3x_6=0\}$	0
	123+456+147+257+367	$\{x_1x_4+x_2x_5+x_3x_6=x_7^2\}$	0	0

n	μ	$\mathcal{R}_1$	$\mathcal{R}_2=\mathcal{R}_3$	$\mathcal{R}_4=\mathcal{R}_5$
8	147+257+367+358	$\mathbb{C}^8$	$\{x_7=0\}$	$\{x_3=x_5=x_7=x_8=0\}\cup\{x_1=x_3=x_4=x_5=x_7=0\}$
	456+147+257+367+358	$\mathbb{C}^8$	$\{x_5=x_7=0\}$	$\{x_3=x_4=x_5=x_7=x_1x_8+x_6^2=0\}$
	123+456+147+358	$\mathbb{C}^8$	$\{x_1=x_5=0\}\cup\{x_3=x_4=0\}$	$\{x_1=x_3=x_4=x_5=x_2x_6+x_7x_8=0\}$
	123+456+147+257+358	$\mathbb{C}^8$	$\{x_1=x_5=0\}\cup\{x_3=x_4=x_5=0\}$	$\{x_1=x_2=x_3=x_4=x_5=x_7=0\}$
	123+456+147+257+367+358	$\mathbb{C}^8$	$\{x_3=x_5=x_1x_4-x_7^2=0\}$	$\{x_1=x_2=x_3=x_4=x_5=x_6=x_7=0\}$
	147+268+358$^{\#}$	$\mathbb{C}^8$	$\{x_1=x_4=x_7=0\}\cup\{x_8=0\}$	$\{x_1=x_4=x_7=x_8=0\}\cup\{x_2=x_3=x_5=x_6=x_8=0\}$
	147+257+268+358	$\mathbb{C}^8$	$L_1\cup L_2\cup L_3$	$L_1\cup L_2$
	456+147+257+268+358	$\mathbb{C}^8$	$C_1\cup C_2$	$L_1\cup L_2$
	147+257+367+268+358	$\mathbb{C}^8$	$L_1\cup L_2\cup L_3\cup L_4$	$L'_1\cup L'_2\cup L'_3$
	456+147+257+367+268+358	$\mathbb{C}^8$	$C_1\cup C_2\cup C_3$	$L_1\cup L_2\cup L_3$
	123+456+147+268+358	$\mathbb{C}^8$	$C_1\cup C_2$	L
	123+456+147+257+268+358	$\mathbb{C}^8$	$\{f_1=\cdots=f_{20}=0\}$	0
	123+456+147+257+367+268+358	$\mathbb{C}^8$	$\{g_1=\cdots=g_{20}=0\}$	0

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex.
- The fundamental group $\pi = \pi_1(X, x_0)$ is a finitely presented group, with abelianization $\pi_{\text{ab}} \cong H_1(X, \mathbb{Z})$.
- The group-algebra $R = \mathbb{C}[\pi_{\text{ab}}]$ is the coordinate ring of the character group, $\text{Char}(X) = \text{Hom}(\pi, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^n \times \text{Tors}(\pi_{\text{ab}})$, where $n = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \text{Char}(X) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- Away from 1, we have that $\mathcal{V}^1_s(X) = V(E_s(A_\pi))$, the zero-set of the ideal of codimension s minors of the Alexander matrix of abelianized Fox derivatives of the relators of π.

THE ALEXANDER POLYNOMIAL

- The group-algebra $\mathbb{C}[\pi_{\mathrm{ab}}/\operatorname{Tors}(\pi_{\mathrm{ab}})]$ is isomorphic to $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the coordinate ring of $\operatorname{Char}^0(X) \cong (\mathbb{C}^\times)^n$.
- The *Alexander polynomial* Δ_X is the gcd of $E_1(A_\pi \otimes_R \Lambda)$.
- Dimca–Papadima–S. (2011): The zero-set $V(\Delta_X)$ coincides (away from 1) with the union of all codimension 1 irreducible components of $\mathcal{V}_1^1(X) \cap \operatorname{Char}^0(X)$.

EXAMPLE

Let K be a knot in S^3. Its complement, X, is a homology circle. The Alexander polynomial, $\Delta = \Delta_X$, satisfies $\Delta(1) = \pm 1$, and so $1 \notin V(\Delta)$. On the other hand, $\mathcal{V}_1^1(X) = V(\Delta) \cup \{1\}$.

Tangent cones and exponential maps

- The map $\exp\colon \mathbb{C}^n \to (\mathbb{C}^\times)^n$, $(z_1, \dots, z_n) \mapsto (e^{z_1}, \dots, e^{z_n})$ is a homomorphism taking 0 to 1.
- For a Zariski-closed subset $W = V(I)$ inside $(\mathbb{C}^\times)^n$, define:
  - The *tangent cone* at 1 to W as $\mathrm{TC}_1(W) = V(\mathrm{in}(I))$.
  - The *exponential tangent cone* at 1 to W as

$$\tau_1(W) = \{z \in \mathbb{C}^n \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}$$

- These sets are homogeneous subvarieties of $\mathbb{C}^n$, which depend only on the analytic germ of W at 1.
- Both commute with finite unions and arbitrary intersections.
- $\tau_1(W) \subseteq \mathrm{TC}_1(W)$.
  - $=$ if all irred components of W are subtori.
  - $\neq$ in general.
- $\tau_1(W)$ is a finite union of rationally defined subspaces.

THE TANGENT CONE THEOREM

- The *resonance varieties* of a space X are the jump loci $\mathcal{R}^i_d(X) \subset H^1(X,\mathbb{C}) = \mathbb{C}^n$ associated to the algebra $A = H^*(X,\mathbb{C})$.
- We also have the characteristic varieties $\mathcal{V}^i_s(X) \subset \operatorname{Char}(X)$. Let $\mathcal{W}^i_s(X) := \mathcal{V}^i_s(X) \cap \operatorname{Char}^0(X) = (\mathbb{C}^\times)^n$.
- (Libgober 2002)
$$\operatorname{TC}_1(\mathcal{W}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$
- Thus,
$$\tau_1(\mathcal{W}^i_s(X)) \subseteq \operatorname{TC}_1(\mathcal{W}^i_s(X)) \subseteq \mathcal{R}^i_s(X).$$
- (DPS 2009/DP 2014) If X is formal, then
$$\tau_1(\mathcal{W}^i_s(X)) = \operatorname{TC}_1(\mathcal{W}^i_s(X)) = \mathcal{R}^i_s(X).$$

A Tangent Cone theorem for 3-manifolds

- Let M be a closed, orientable, 3-dimensional manifold.
- C. McMullen (2000): Let I be the augmentation ideal of Λ. Then

$$E_1(M) = \begin{cases} (\Delta_M) & \text{if } b_1(M) \leqslant 1, \\ I^2 \cdot (\Delta_M) & \text{if } b_1(M) \geqslant 2. \end{cases}$$

- It follows that $\mathcal{W}^1_1(M) = V(\Delta_M)$, at least away from 1.
- Using the previous discussion, as well as work of Turaev (2002), we obtain:

Theorem

Suppose $b_1(M)$ is odd and μ_M is generic. Then

$$\operatorname{TC}_1(\mathcal{W}^1_1(M)) = \mathcal{R}^1_1(M).$$

- If $b_1(M)$ is even, the conclusion of the theorem may or may not hold:
  - Let $M = S^1 \times S^2 \# S^1 \times S^2$; then $\mathcal{V}^1_1(M) = \operatorname{Char}(M) = (\mathbb{C}^\times)^2$, and so $\operatorname{TC}_1(\mathcal{V}^1_1(M)) = \mathcal{R}^1_1(M) = \mathbb{C}^2$.
  - Let M be the Heisenberg nilmanifold; then $\operatorname{TC}_1(\mathcal{V}^1_1(M)) = \{0\}$, whereas $\mathcal{R}^1_1(M) = \mathbb{C}^2$.
- If M is not formal, the first half of the Tangent Cone theorem may fail to hold, i.e., $\tau_1(\mathcal{V}^1_1(M)) \nsubseteq \operatorname{TC}_1(\mathcal{V}^1_1(M))$.
  - Let M be a closed, orientable 3-manifold with $b_1 = 7$ and $\mu = e_1e_3e_5 + e_1e_4e_7 + e_2e_5e_7 + e_3e_6e_7 + e_4e_5e_6$. Then μ is generic and $\operatorname{Pf}(\mu) = (x_5^2 + x_7^2)^2$. Hence, $\mathcal{R}^1_1(M) = \{x_5^2 + x_7^2 = 0\}$ splits as a union of two hyperplanes over $\mathbb{C}$, but not over $\mathbb{Q}$.

DUALITY AND ABELIAN DUALITY SPACES

- Let X be a path-connected space, having the homotopy type of a finite-type CW-complex. Set $\pi = \pi_1(X)$.
- Bieri and Eckmann (1978): X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $D := H^n(X, \mathbb{Z}\pi)$ is non-zero and torsion-free.
- Then $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$, for any $\mathbb{Z}\pi$-module A.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a PD space.
- Denham–S.–Yuzvinsky (2016): X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\text{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\text{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\text{ab}})$ be the dualizing $\mathbb{Z}\pi_{\text{ab}}$-module. Given any $\mathbb{Z}\pi_{\text{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.

PROPAGATION OF JUMP LOCI

THEOREM (DSY 2016/2017)

Let X be an abelian duality space of dimension n. Then:

- *If $H^i(X, \mathbb{C}_\rho) \neq 0$, then $H^j(X, \mathbb{C}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.*
- *The characteristic varieties propagate: $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.*
- *$b_1(X) \geqslant n-1$.*
- *If $n \geqslant 2$, then $b_i(X) \neq 0$, for all $0 \leqslant i \leqslant n$.*
- *If, moreover, X is formal, then the resonance varieties propagate: $\mathcal{R}_1^1(X) \subseteq \cdots \subseteq \mathcal{R}_1^n(X)$.*

- Let M be a compact, connected, orientable smooth manifold of dimension n. By Poincaré duality, $\mathcal{R}_1^n(M) = \{0\}$.
- On the other hand, if $n = 3$ and $b_1(M)$ is even and non-zero, then $\mathcal{R}_1^1(M) = H^1(M, \mathbb{C})$.
- Hence, such a 3-manifold M is *not* an abelian duality space.

Arrangements of smooth hypersurfaces

Theorem (Denham–S. 2017)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

- *Components of the boundary $D = Y \backslash U$ form an arrangement of smooth hypersurfaces $\mathcal{A}$;*
- *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

Consequently, the characteristic varieties of such "recursively Stein" hypersurface complements propagate.

Theorem (DSY/DS)

Suppose that $\mathcal{A}$ is one of the following:

- *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
- *A non-empty elliptic arrangement in E^n;*
- *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

As a consequence, the characteristic varieties propagate for all linear, elliptic and toric arrangements. The formality of linear and toric arrangement complements implies that their resonance varieties propagate, as well.