FUNDAMENTAL GROUPS IN COMPLEX GEOMETRY AND 3-DIMENSIONAL TOPOLOGY

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FUNDAMENTAL GROUPS OF MANIFOLDS

- Every finitely presented group π can be realized as $\pi = \pi_1(M)$, for some smooth, compact, connected manifold M^n of dim $n \geqslant 4$.
- M^n can be chosen to be orientable.
- If n even, $n \geqslant 4$, then M^n can be chosen to be symplectic (Gompf).
- If n even, $n \geqslant 6$, then M^n can be chosen to be complex (Taubes).
- Requiring that $n = 3$ puts severe restrictions on the (closed) 3-manifold group $\pi = \pi_1(M^3)$.

KÄHLER GROUPS & 3-MANIFOLD GROUPS

- A *Kähler manifold* is a compact, connected, complex manifold, with a Hermitian metric h such that $\omega = \mathrm{im}(h)$ is a closed 2-form.
- Examples: smooth, complex projective varieties.
- If M is a Kähler manifold, $\pi = \pi_1(M)$ is called a *Kähler group*.
- This also puts strong restrictions on π, e.g.:
  - $b_1(\pi)$ is even (Hodge theory)
  - π is 1-formal: Malcev Lie algebra $\mathfrak{m}(\pi)$ is quadratic (DGMS 1975)
  - π cannot split non-trivially as a free product (Gromov 1989)
- π finite $\Rightarrow$ π projective group (Serre 1958).

QUESTION (DONALDSON–GOLDMAN 1989)

Which 3-manifold groups are Kähler groups?

Reznikov (2002) gave a partial solution.

THEOREM (DIMCA–S. 2009)

Let π be the fundamental group of a closed 3-manifold. Then π is a Kähler group $\Longleftrightarrow$ π is a finite subgroup of $O(4)$, acting freely on S^3.

Alternative proofs have since been given by Kotschick (2012) and by Biswas, Mj and Seshadri (2012).

THEOREM (FRIEDL–S. 2014)

Let N be a 3-manifold with non-empty, toroidal boundary. If $\pi_1(N)$ is a Kähler group, then $N \cong S^1 \times S^1 \times I$.

Since then, Kotschick has generalized this result, by dropping the toroidal boundary assumption: If $\pi_1(N)$ is an infinite Kähler group, then $\pi_1(N)$ is a surface group.

QUASI-PROJECTIVE GROUPS & 3-MANIFOLD GROUPS

- A group π is called a *quasi-projective group* if $\pi = \pi_1(M \backslash D)$, where M is a smooth, projective variety and D is a divisor.
- Qp groups are finitely presented. The class of qp groups is closed under direct products and passing to finite-index subgroups.
- For a qp group π,
  - $b_1(\pi)$ can be arbitrary (e.g., the free groups F_n).
  - π may be non-1-formal (e.g., the Heisenberg group).
  - π can split as a non-trivial free product.
- Subclass: fundamental groups of complements of hypersurfaces in $\mathbb{CP}^n$, or, equivalently, fundamental groups of complements of plane algebraic curves.
- Such groups are 1-formal.

QUESTION (DIMCA–S. 2009)

Which 3-manifold groups are quasi-projective groups?

THEOREM (DIMCA–PAPADIMA–S. 2011)

Let π be the fundamental group of a closed, orientable 3-manifold. Assume π is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(X))$, *for some quasi-projective manifold* X.
2. $\mathfrak{m}(\pi) \cong \mathfrak{m}(\pi_1(N))$, *where* N *is either* S^3, $\#^n S^1 \times S^2$, *or* $S^1 \times \Sigma_g$.

Joint work with Stefan Friedl (2014)

THEOREM

Let N be a 3-mfd with empty or toroidal boundary. If $\pi_1(N)$ is a quasi-projective group, then all prime components of N are graph manifolds.

In particular, the fundamental group of a hyperbolic 3-manifold with empty or toroidal boundary is never a qp-group.

ALEXANDER POLYNOMIALS

- Let H be a finitely generated, free abelian group.
- Let M be a finitely generated module over $\Lambda = \mathbb{Z}[H]$. Pick a presentation $\Lambda^p \xrightarrow{\alpha} \Lambda^s \longrightarrow M \longrightarrow 0$ with $p \geqslant s$.
- Let $E_i(M)$ be the ideal of minors of size $s - i$ of α, and set

$$\operatorname{ord}^i(M) := \gcd(E_i(M)) \in \Lambda$$

(well-defined up to units in Λ).

- Write $r = \operatorname{rank}(M)$, and set

$$\Delta_M^k := \begin{cases} \operatorname{ord}^{k-r}(\operatorname{Tors} M) & \text{if } k \geqslant r \\ 0 & \text{if } k < r \end{cases}$$

- Define the *thickness* of M as

$$\operatorname{th}(M) = \dim \operatorname{Newt}(\Delta_M^r).$$

- Let X be a finite, conn. CW-complex. Write $H := H_1(X;\mathbb{Z})/\operatorname{Tors}$.
  - Alexander invariant: $A_X = H_1(X;\mathbb{Z}[H])$.
  - Alexander polynomials: $\Delta_X^k = \operatorname{ord}^k(A_X)$; usual one: $\Delta = \Delta^0$.
  - Set $\operatorname{th}(X) := \operatorname{th}(A_X)$. Note: $\operatorname{th}(X) = \operatorname{th}(\pi_1(X))$.
- Let $\widehat{H} = \operatorname{Hom}(H, \mathbb{C}^*)$ be the character torus. Define hypersurfaces

$$V(\Delta_X^k) = \{\rho \in \widehat{H} \mid \Delta_X^k(\rho) = 0\}.$$

- If $X = S^3 \setminus K$, then Δ_X is the classical Alexander polynomial of K, and $V(\Delta_X^k) \subset \mathbb{C}^*$ is the set of roots of Δ_X, of multiplicity at least k.
- Also define the (degree 1) *characteristic varieties* of X as

$$\mathcal{V}_k(X) = \{\rho \in \widehat{H} \mid \dim H_1(X, \mathbb{C}_\rho) \geqslant k\},$$

  where $\mathbb{C}_\rho = \mathbb{C}$, viewed as a module over $\mathbb{Z}H$, via $g \cdot x = \rho(g)x$.
- We then have: $\mathcal{V}_k(X) \setminus \{1\} = V(E_{k-1}(A_X)) \setminus \{1\}$.

Let $\check{\mathcal{V}}_k(X)$ be the union of all codim 1 irreducible components of $\mathcal{V}_k(X)$.

LEMMA (DPS08 for $k=1$, FS14 for $k>1$)

1. $\Delta_X^{k-1} = 0$ *if and only if* $\mathcal{V}_k(X) = \widehat{H}$, *in which case* $\check{\mathcal{V}}_k(X) = \emptyset$.
2. *Suppose* $b_1(X) \geqslant 1$ *and* $\Delta_X^{k-1} \neq 0$. *Then at least away from* 1,

$$\check{\mathcal{V}}_k(X) = V(\Delta_X^{k-1}).$$

THEOREM (DPS, FS)

Suppose $b_1(X) \geqslant 2$. *Then* $\Delta_X^{k-1} \doteq \text{const}$ *if and only if* $\check{\mathcal{V}}_k(X) = \emptyset$. *Otherwise, the following are equivalent:*

1. *The Newton polytope of* Δ_X^{k-1} *is a line segment.*
2. *All irreducible components of* $\check{\mathcal{V}}_k(X)$ *are parallel, codim* 1 *subtori of* $\widehat{H}$.

The next theorem is due to Arapura (1997), with improvements by DPS (2008, 2009) and Artal-Bartolo, Cogolludo, Matei (2013).

THEOREM

Let π be a quasi-projective group. Then, for each $k \geqslant 1$,

- *The irreducible components of $\mathcal{V}_k(\pi)$ are (possibly torsion-translated) subtori of the character torus $\widehat{H}$.*
- *Any two distinct components of $\mathcal{V}_k(\pi)$ meet in a finite set.*

Using this theorem, we prove

THEOREM (DPS08 FOR $k = 0$, FS14 FOR $k > 0$)

Let π be a quasi-projective group, and assume $b_1(\pi) \neq 2$. Then, for each $k \geqslant 0$, the polynomial Δ_π^k is either zero, or the Newton polytope of Δ_π^k is a point or a line segment. In particular, $\operatorname{th}(\pi) \leqslant 1$.

THURSTON NORM AND ALEXANDER NORM

- Let N be a 3-manifold with either empty or toroidal boundary.
- A class $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ is *fibered* if there exists a fibration $p\colon N \to S^1$ such that $p_*\colon \pi_1(N) \to \mathbb{Z}$ coincides with ϕ.
- Given a surface Σ with connected components $\Sigma_1,\dots,\Sigma_s$, put $\chi_-(\Sigma) = \sum_{i=1}^{s} \max\{-\chi(\Sigma_i), 0\}$.
- *Thurston norm*: $\|\phi\|_T = \min\{\chi_-(\Sigma)\}$, where Σ runs through all the properly embedded surfaces dual to ϕ.
- $\| - \|_T$ defines a (semi)norm on $H^1(N;\mathbb{Z})$, which can be extended to a (semi)norm $\| - \|_T$ on $H^1(N;\mathbb{Q})$.
- The unit norm ball, $B_T = \{\phi \in H^1(N;\mathbb{Q}) \mid \|\phi\|_T \leqslant 1\}$, is a rational polyhedron with finitely many sides, symmetric in the origin.

- The set of fibered classes form a cone on certain open, top-dimensional faces of B_T, called the *fibered faces* of B_T.
- Two faces F and G are *equivalent* if $F = \pm G$. Clearly, F is fibered if and only if $-F$ is fibered.

We say $\phi \in H^1(N;\mathbb{Q})$ is *quasi-fibered* if it lies on the boundary of a fibered face of B_T. Results of Stallings (1962) and Gabai (1983) imply

COROLLARY (FS14)

Let $p\colon N' \to N$ be a finite cover. Then:

1. $\phi \in H^1(N;\mathbb{Q})$ *quasi-fibered* $\Rightarrow p^*(\phi) \in H^1(N';\mathbb{Q})$ *quasi-fibered.*
2. *Pull-backs of inequivalent faces of the Thurston norm ball of N lie on inequivalent faces of the Thurston norm ball of N'.*

- Let $\Delta_N = \sum_{h \in H} a_h h \in \mathbb{Z}[H]$ be the Alexander polynomial of N.
- Define a (semi)norm $\| - \|_A$ on $H^1(N; \mathbb{Q})$ by

$$\|\phi\|_A := \max\{\phi(a_h) - \phi(a_g) \mid g, h \in H \text{ with } a_g \neq 0 \text{ and } a_h \neq 0\}.$$

THEOREM (MCMULLEN 2002)

Let N be a 3-manifold with empty or toroidal boundary and such that $b_1(N) \geqslant 2$. Then $\|\phi\|_A \leqslant \|\phi\|_T$, for any $\phi \in H^1(N; \mathbb{Q})$. Furthermore, equality holds for any quasi-fibered class.

COROLLARY (FS14)

Let N be a 3-manifold with empty or toroidal boundary.

- *If there is a fibration $F \to N \to S^1$ with $\chi(F) < 0$, then $\mathrm{th}(N) \geqslant 1$.*
- *If N has at least two non-equivalent fibered faces, then $\mathrm{th}(N) \geqslant 2$.*

THE RFRS PROPERTY

DEFINITION (AGOL 2008)

A group π is called *residually finite rationally solvable (RFRS)* if there is a filtration $\pi = \pi_0 \geqslant \pi_1 \geqslant \pi_2 \geqslant \cdots$ such that $\bigcap_i \pi_i = \{1\}$, and

- Each group π_i is a normal, finite-index subgroup of π.
- Each map $\pi_i \to \pi_i / \pi_{i+1}$ factors through $\pi_i \to H_1(\pi_i; \mathbb{Z}) / \operatorname{Tors}$.

E.g., free groups and surface groups are RFRS.

THEOREM (AGOL 2008)

Let N be an irreducible 3-manifold such that $\pi_1(N)$ is virtually RFRS. Let $\phi \in H^1(N; \mathbb{Q})$ be a non-fibered class. There exists then a finite cover $p\colon N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Q})$ is quasi-fibered.

Assume N is an irreducible 3-manifold with empty or toroidal boundary.

THEOREM (AGOL, WISE, PRZYTYCKI– WISE, ...)

If N is not a closed graph manifold, then $\pi_1(N)$ is virtually RFRS.

COROLLARY

If N is not a closed graph manifold, then N is virtually fibered.

THEOREM (AGOL, WISE, ...)

Suppose N is neither $S^1 \times D^2$, nor $T^2 \times I$, nor finitely cover by a torus bundle. Then, $\forall k \in \mathbb{N}$, there is a finite cover $N' \to N$ s.t. $b_1(N') \geqslant k$.

THEOREM

Suppose N is not a graph manifold. Given any $k \in \mathbb{N}$, there exists a finite cover $N' \to N$ such that the Thurston norm ball of N' has at least k non-equivalent fibered faces.

QUASI-PROJECTIVE 3-MANIFOLD GROUPS

THEOREM (FS14)

Suppose N is not a graph manifold. There exists then a finite cover $N' \to N$ with $\mathrm{th}(N') \geqslant 2$ and $b_1(N') \geqslant 3$.

PROOF.

- Since N is not a graph manifold, it admits finite covers with arbitrarily large first Betti numbers.
- We can thus assume that $b_1(N) \geqslant 3$.
- There exists a finite cover $N' \to N$ such that the Thurston norm ball of N' has at least 2 non-equivalent fibered faces.
- A transfer argument shows that $b_1(N') \geqslant b_1(N) \geqslant 3$.
- Hence, $\mathrm{th}(N') \geqslant 2$.

□

We can now prove our theorem in the case when N is irreducible.

THEOREM (FS14)

Let N be an irreducible 3-manifold with empty or toroidal boundary. If N is not a graph manifold, then $\pi_1(N)$ is not a quasi-projective group.

PROOF.

- Suppose $\pi_1(N)$ is a qp group.
- We know there is a finite cover $N' \to N$ with $\operatorname{th}(N') \geqslant 2$ and $b_1(N') \geqslant 3$.
- On the other hand, $\pi_1(N')$ is also a qp group.
- Hence, either $b_1(N') = 2$, or $\operatorname{th}(N') \leqslant 1$.
- This is a contradiction.

□

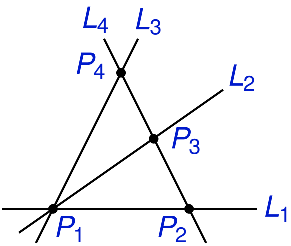
The case when N has several prime factors is more complicated, but can be handled with similar techniques.

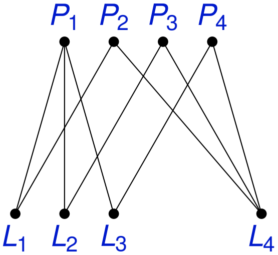
PLANE ALGEBRAIC CURVES

- Let $\mathcal{C} \subset \mathbb{CP}^2$ be a plane algebraic curve, defined by a homogeneous polynomial $f \in \mathbb{C}[z_1, z_2, z_3]$.
- Zariski commissioned Van Kampen to find a presentation for the fundamental group of the complement, $U(\mathcal{C}) = \mathbb{CP}^2 \backslash \mathcal{C}$.
- Zariski noticed that $\pi = \pi_1(U)$ is *not* fully determined by the combinatorics of $\mathcal{C}$, but depends on the position of its singularities.
- He asked whether π is *residually finite*, i.e., whether the map to its profinite completion, $\pi \to \widehat{\pi} =: \pi^{\mathrm{alg}}$, is injective.

LINE ARRANGEMENTS

- Let $\mathcal{A}$ be an *arrangement of lines* in $\mathbb{CP}^2$, defined by a polynomial $f = \prod_{L \in \mathcal{A}} f_L$, with f_L linear forms so that $L = \mathbb{P}(\ker(f_L))$.
- The combinatorics of $\mathcal{A}$ is encoded in the *intersection poset*, $\mathcal{L}(\mathcal{A})$, with $\mathcal{L}_1(\mathcal{A}) = \{\text{lines}\}$ and $\mathcal{L}_2(\mathcal{A}) = \{\text{intersection points}\}$.





- The group $\pi = \pi_1(U(\mathcal{A}))$ has a finite presentation with
  - Meridional generators $x_1, \dots, x_n$, where $n = |\mathcal{A}|$, and $\prod x_i = 1$.
  - Commutator relators $x_i \alpha_j (x_i)^{-1}$, where $\alpha_1, \dots \alpha_s \in P_n \subset \mathrm{Aut}(F_n)$, and $s = |\mathcal{L}_2(\mathcal{A})|$.

- Let $\pi/\gamma_k(\pi)$ be the $(k-1)^{\text{th}}$ nilpotent quotient of π. Then:
  - $\pi_{\text{ab}} = \pi/\gamma_2$ equals $\mathbb{Z}^{n-1}$.
  - π/γ_3 is determined by $L(\mathcal{A})$.
  - π/γ_4 (and thus, π) is *not* determined by $L(\mathcal{A})$. (Rybnikov).

THEOREM (S. 2011)

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, with group $\pi = \pi_1(U(\mathcal{A}))$. The following are equivalent:

1. *π is a Kähler group.*
2. *π is a free abelian group of even rank.*
3. *$\mathcal{A}$ consists of an odd number of lines in general position.*

THEOREM (DPS 2009)

Let Γ be a finite simple graph, and A_Γ the corresponding RAAG. Then:

1. *A_Γ is a quasi-projective group if and only if Γ is a complete multipartite graph $K_{n_1,\dots,n_r} = \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$, in which case $A_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$.*
2. *A_Γ is a Kähler group if and only if Γ is a complete graph K_{2m}, in which case $G_\Gamma = \mathbb{Z}^{2m}$.*

THEOREM (S. 2011)

Let $\pi = \pi_1(U(\mathcal{A}))$. *The following are equivalent:*

1. π *is a RAAG.*
2. π *is a finite direct product of finitely generated free groups.*
3. $\mathcal{G}(\mathcal{A})$ *is a forest.*

Here $\mathcal{G}(\mathcal{A})$ is the 'multiplicity' graph, with

- vertices: points $P \in \mathcal{L}_2(\mathcal{A})$ with multiplicity at least 3;
- edges: $\{P, Q\}$ if $P, Q \in L$, for some $L \in \mathcal{A}$.

THE RFRp PROPERTY

Joint work with Thomas Koberda (2016)

Let G be a finitely generated group and let p be a prime.

We say that G is *residually finite rationally p* if there exists a sequence of subgroups $G = G_0 > \cdots > G_i > G_{i+1} > \cdots$ such that

1. $G_{i+1} \lhd G_i$.
2. $\bigcap_{i \geqslant 0} G_i = \{1\}$.
3. G_i / G_{i+1} is an elementary abelian p-group.
4. $\ker(G_i \to H_1(G_i, \mathbb{Q})) < G_{i+1}$.

Remarks:

- We may assume each $G_i \lhd G$.
- Compare with Agol's RFRS property, where he only assumes G_i / G_{i+1} is finite.

- G RFRp $\Rightarrow$ residually p $\Rightarrow$ residually finite and residually nilpotent.
- G RFRp $\Rightarrow$ G RFRS $\Rightarrow$ torsion-free.
- The class of RFRp groups is closed under the following operations:
  - Taking subgroups.
  - Finite direct products.
  - Finite free products.
- The following groups are RFRp, for all p:
  - Finitely generated free groups.
  - Closed, orientable surface groups.
  - Right-angled Artin groups.

A COMBINATION THEOREM

THEOREM (KS16)

Fix a prime p. Let $X = X_\Gamma$ be a finite graph of connected, finite CW-complexes with vertex spaces $\{X_v\}_{v\in V(\Gamma)}$ and edge spaces $\{X_e\}_{e\in E(\Gamma)}$ satisfying the following conditions:

1. *For each $v \in V(\Gamma)$, the group $\pi_1(X_v)$ is RFRp.*
2. *For each $v \in V(\Gamma)$, the RFRp topology on $\pi_1(X)$ induces the RFRp topology on $\pi_1(X_v)$ by restriction.*
3. *For each $e \in E(\Gamma)$ and each $v \in e$, the subgroup $\phi_{e,v}(\pi_1(X_e))$ of $\pi_1(X_v)$ is closed in the RFRp topology on $\pi_1(X_v)$.*

Then $\pi_1(X)$ is RFRp.

BOUNDARY MANIFOLDS

- Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$, and let N be a regular neighborhood of $\bigcup_{L\in\mathcal{A}} L$.
- The *boundary manifold* of $\mathcal{A}$ is $M = \partial N$, a compact, orientable, smooth manifold of dimension 3.

EXAMPLE

Let $\mathcal{A}$ be a pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1^n - z_2^n$.
If $n = 1$, then $M = S^3$. If $n > 1$, then $M = \sharp^{n-1} S^1 \times S^2$.

EXAMPLE

Let $\mathcal{A}$ be a near-pencil of n lines in $\mathbb{CP}^2$, defined by $f = z_1(z_2^{n-1} - z_3^{n-1})$. Then $M = S^1 \times \Sigma_{n-2}$, where $\Sigma_g = \sharp^g S^1 \times S^1$.

- M is a graph-manifold M_Γ, where Γ is the incidence graph of $\mathcal{A}$, with $V(\Gamma) = L_1(\mathcal{A}) \cup L_2(\mathcal{A})$ and $E(\Gamma) = \{(L, P) \mid P \in L\}$.
- For each $v \in V(\Gamma)$, there is a vertex manifold $M_v = S^1 \times S_v$, with $S_v = S^2 \setminus \bigcup_{\{v,w\} \in E(\Gamma)} D^2_{v,w}$.
- Vertex manifolds are glued along edge manifolds $M_e = S^1 \times S^1$ via flips.
- The boundary manifold of a line arrangement in $\mathbb{C}^2$ is defined as $M = \partial N \cap D^4$, for some sufficiently large 4-ball D^4.

THEOREM (KS16)

If M is the boundary manifold of a line arrangement in $\mathbb{C}^2$, then $\pi_1(M)$ is RFRp, for all primes p.

CONJECTURE (KS)

Arrangement groups are RFRp, for all primes p.

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