ABELIAN DUALITY SPACES

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The following notion is due to Bieri and Eckmann (1978).

- Let X be a path-connected space with fundamental group $\pi = \pi_1(X)$.
- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then X is a Poincaré duality space.
- If $X = K(\pi, 1)$ is a duality space, then π is a *duality group*.

In (Denham–S.–Yuzvinsky, 2017) we introduce an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\text{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\text{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\text{ab}})$ be the dualizing $\mathbb{Z}\pi_{\text{ab}}$-module. Given any $\mathbb{Z}\pi_{\text{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- If $X = K(\pi, 1)$ is an abelian duality space, then π is an *abelian duality group*.
- Finitely generated free groups F_n are abelian duality groups.
- Surface groups $\pi_1(\Sigma_g)$ with $g \geqslant 2$ are (Poincaré) duality groups, but not abelian duality groups.
- Let $H = \langle x_1, \dots, x_4 \mid x_1^{-2} x_2 x_1 x_2^{-1}, \dots, x_4^{-2} x_1 x_4 x_1^{-1} \rangle$ be Higman's acyclic group, and let $G = \mathbb{Z}^2 * H$. Then G is an abelian duality group (of dimension 2), but not a duality group.

Theorem (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n-1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.

Theorem (Denham–S. 2017)

Let U be a connected, smooth, complex quasi-projective variety of dimension n. Suppose U has a smooth compactification Y for which

1. *Components of $Y \backslash U$ form an arrangement of hypersurfaces $\mathcal{A}$;*
2. *For each submanifold X in the intersection poset $L(\mathcal{A})$, the complement of the restriction of $\mathcal{A}$ to X is a Stein manifold.*

Then U is both a duality space and an abelian duality space of dimension n.

THEOREM (DS17)

Suppose that $\mathcal{A}$ is one of the following:

- *An affine-linear arrangement in $\mathbb{C}^n$, or a hyperplane arrangement in $\mathbb{CP}^n$;*
- *A non-empty elliptic arrangement in E^n;*
- *A toric arrangement in $(\mathbb{C}^*)^n$.*

Then the complement $M(\mathcal{A})$ is both a duality space and an abelian duality space of dimension $n-r$, $n+r$, and n, respectively, where r is the corank of the arrangement.

This theorem extends several previous results:

1. Davis, Januszkiewicz, Leary, and Okun (2011);
2. Levin and Varchenko (2012);
3. Davis and Settepanella (2013), Esterov and Takeuchi (2014).

Liu, Maxim, and Wang (2017) proved that very affine varieties are abelian duality spaces.

- Let X be a connected, finite-type CW-complex. Then $\pi = \pi_1(X)$ is a finitely presented group, with $\pi_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.
- The ring $R = \mathbb{C}[\pi_{\mathrm{ab}}]$ is the coordinate ring of the character group, $\mathrm{Char}(X) = \mathrm{Hom}(\pi, \mathbb{C}^*) \cong (\mathbb{C}^*)^r \times \mathrm{Tors}(\pi_{\mathrm{ab}})$, where $r = b_1(X)$.
- The *characteristic varieties* of X are the homology jump loci

$$\mathcal{V}^i_s(X) = \{\rho \in \mathrm{Char}(X) \mid \dim H_i(X, \mathbb{C}_\rho) \geqslant s\}.$$

- These varieties are homotopy-type invariants of X, with $\mathcal{V}^1_s(X)$ depending only on $\pi = \pi_1(X)$.
- Set $\mathcal{V}_1(\pi) := \mathcal{V}^1_1(K(\pi, 1))$; then $\mathcal{V}_1(\pi) = \mathcal{V}_1(\pi/\pi'')$.

EXAMPLE

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be a Laurent polynomial, $f(1) = 0$. There is then a finitely presented group π with $\pi_{\mathrm{ab}} = \mathbb{Z}^n$ such that $\mathcal{V}_1(\pi) = \mathbf{V}(f)$.

THEOREM (DSY)

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{C}^*$ satisfies $H^i(X, \mathbb{C}_\rho) \neq 0$, then $H^j(X, \mathbb{C}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

COROLLARY

Let X be an abelian duality space of dimension n. Then the characteristic varieties propagate, i.e., $\mathcal{V}_1^1(X) \subseteq \cdots \subseteq \mathcal{V}_1^n(X)$.

- Let L be a simplicial complex on vertex set $\mathsf{V} = \{v_1, \dots, v_m\}$.
- Let $T_L = \mathcal{Z}_L(S^1, *)$ be the subcomplex of T^m obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a connected CW-complex, of dimension $\dim L + 1$.
- $H^*(T_L, \Bbbk)$ is the exterior Stanley–Reisner ring

$$\Bbbk\langle L\rangle = \bigwedge V^*/(v_\sigma^* \mid \sigma \notin L),$$

  where $\Bbbk = \mathbb{Z}$ or a field, V is the free $\Bbbk$-module on V, and $V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$, while $v_\sigma^* = v_{i_1}^* \cdots v_{i_s}^*$ for $\sigma = \{i_1, \dots, i_s\}$.

- The group $\pi_\Gamma := \pi_1(T_L, *)$ is the right-angled Artin group (RAAG) associated to the graph $\Gamma := L^{(1)} = (\mathsf{V}, \mathsf{E})$,

$$\pi_\Gamma = \langle v \in V \mid [v, w] = 1 \text{ if } \{v, w\} \in E\rangle.$$

- Moreover, $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.

A simplicial complex L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\operatorname{lk}(\sigma)$ is concentrated in degree $\dim L - |\sigma|$ and is torsion-free.

THEOREM (N. BRADY–MEIER 2001, JENSEN–MEIER 2005)

A RAAG π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

THEOREM (DSY17)

A toric complex T_L is an abelian duality space (of dimension $\dim L + 1$) if and only if L is Cohen-Macaulay.

- The Bestvina–Brady group associated to a graph Γ is defined as $N_\Gamma = \ker(\nu\colon \pi_\Gamma \to \mathbb{Z})$, where $\nu(v) = 1$, for each $v \in V(\Gamma)$.
- A counterexample to either the Eilenberg–Ganea conjecture or the Whitehead conjecture can be constructed from these groups.
- The cohomology ring $H^*(N_\Gamma, \Bbbk)$ was computed by Papadima–S. (2007) and Leary–Saadetoğlu (2011).
- The jump loci $\mathcal{V}^1_1(N_\Gamma, \Bbbk)$ were computed in PS07.

THEOREM (DAVIS–OKUN 2012)

Suppose Δ_Γ is acyclic. Then N_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay.

THEOREM (DSY17)

A Bestvina–Brady group N_Γ is an abelian duality group if and only if Δ_Γ is acyclic and Cohen–Macaulay.

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