Resonance varieties of differential graded algebras

Alex Suciu

Northeastern University

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Commutative differential graded algebras

- Let $A = (A^\bullet, d)$ be a commutative, differential graded algebra over a field $\Bbbk$.
  - $A = \bigoplus_{i \geqslant 0} A^i$, where A^i are $\Bbbk$-vector spaces.
  - The multiplication $\cdot\colon A^i \otimes A^j \to A^{i+j}$ is graded-commutative, i.e., $ab = (-1)^{|a||b|} ba$ for all homogeneous a and b.
  - The differential $d\colon A^i \to A^{i+1}$ satisfies the graded Leibniz rule, i.e., $d(ab) = d(a)b + (-1)^{|a|} ad(b)$.
- $H^\bullet(A)$ inherits an algebra structure from A.
- A cdga morphism $\varphi\colon A \to B$ is both an algebra map and a cochain map. Hence, it induces a morphism $\varphi^*\colon H^\bullet(A) \to H^\bullet(B)$.
- φ is a quasi-isomorphism if φ^* is an isomorphism. Likewise, φ is a q-quasi-isomorphism (for some $q \geqslant 1$) if φ^* is an isomorphism in degrees $\leqslant q$ and is injective in degree $q+1$.
- Two cdgas, A and B, are *(q-)equivalent* ($\simeq_q$) if there is a zig-zag of (q-)quasi-isomorphisms connecting A to B.

Maurer–Cartan sets

- Let $\mathrm{MC}(A) = \{a \in A^1 \mid a^2 + d(a) = 0 \in A^2\}$.
- If $\dim_{\Bbbk}(A^1) < \infty$, then $\mathrm{MC}(A)$ is an algebraic subvariety of the affine space A^1, cut out by quadratic and linear equations.
- Examples:
  - If $a^2 + d(a) = 0$ for all $a \in A^1$, then $\mathrm{MC}(A) = A^1$.
  - If $a^2 = 0$, for all $a \in A^1$ (a condition that is always satisfied if $\mathrm{char}(\Bbbk) \neq 2$), then $\mathrm{MC}(A) = Z^1(A)$.
  - If $a^2 = 0$ and A is *connected* (that is, $A^0 = \Bbbk \cdot 1$), then $d\colon A^0 \to A^1$ vanishes (since $d(1) = 0$), and so $\mathrm{MC}(A) = H^1(A)$.
- If $\varphi\colon A \to B$ is a morphism of cdgas, then the linear map $\varphi^1\colon A^1 \to B^1$ restricts to a map $\bar{\varphi}\colon \mathrm{MC}(A) \to \mathrm{MC}(B)$.

- Assume A is of finite type, i.e., $\dim_{\mathbb{k}} A^i < \infty$ for all i.
- For each $a \in \mathrm{MC}(A)$, we have a cochain complex,

$$(A^\bullet, \delta_a^A)\colon\ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots,$$

  with differentials $\delta_a^i\colon A^i \to A^{i+1}$ the $\mathbb{k}$-linear maps given by $\delta_a^i(u) = a \cdot u + d(u)$ for $u \in A^i$.
- Let $b_i(A, a) := \dim_{\mathbb{k}} H^i(A, \delta_a^A)$. Note that $0 \in \mathrm{MC}(A)$ and $\delta_0 = d$; thus, $b_i(A, 0) = b_i(A)$.
- Let $\varphi\colon (A, d_A) \to (B, d_B)$ be a a morphism. For each $a \in \mathrm{MC}(A)$, the map φ induces a chain map, $\varphi_a\colon (A^\bullet, \delta_a^A) \to (B^\bullet, \delta_{\varphi(a)}^B)$.
- In turn, φ_a induces homomorphisms in cohomology,

$$\varphi_a^i\colon H^i\left(A, \delta_a^A\right) \longrightarrow H^i\left(B, \delta_{\bar{\varphi}(a)}^B\right).$$

The Koszul complex of a CDGA

- Let $A = (A^\bullet, d)$ be a connected $\Bbbk$-CDGA with $\dim_\Bbbk A^1 < \infty$.
- Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^\vee$.
- Identify the symmetric algebra $\mathrm{Sym}(A_1)$ with the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$.
- The coordinate ring of the affine variety $\mathrm{MC}(A) \subset A^1$ is the quotient, $S = R/I$, of the ring R by the defining ideal of $\mathrm{MC}(A)$.
- $A^\bullet \otimes S$ is both a free S-module and a bigraded $\Bbbk$-algebra, with product $(a \otimes s)(a' \otimes s') = aa' \otimes ss'$. It is also a $\Bbbk$-CDGA, with differential $d \otimes \mathrm{id}_S$.
- Under the identification $A^1 \otimes A_1 \cong \mathrm{Hom}(A^1, A^1)$, the "canonical element" $\omega_A = \sum_{j=1}^n e_j \otimes x_j$ corresponds to the identity map of A^1.

- Left-multiplication by ω_A defines an endomorphism of $A \otimes S$ of bidegree $(1, 1)$.
- Let $\delta_A \colon A \otimes S \to A \otimes S$ be the S-linear map given by

$$\delta_A = \omega_A + d \otimes \mathrm{id}_S \,.$$

- We have $\delta_A^2 = 0$, and so we get a cochain complex of free S-modules,

$$\cdots \longrightarrow A^i \otimes S \xrightarrow{\delta_A^i} A^{i+1} \otimes S \xrightarrow{\delta_A^{i+1}} A^{i+2} \otimes S \longrightarrow \cdots$$

- $(A^\bullet \otimes S, \delta_A)$ is again a $\Bbbk$-CDGA.
- The specialization of $(A \otimes S, \delta_A)$ at $a \in \mathrm{MC}(A)$ coincides with (A, δ_a).

Alexander invariants

- S-dual chain complex:

$$(A_\bullet \otimes S, \delta^A) : \cdots \longrightarrow A_2 \otimes S \xrightarrow{\delta_2^A} A_1 \otimes S \xrightarrow{\delta_1^A} A_0 \otimes S.$$

- The *Alexander invariants* of CDGA $(A^\bullet, d)$ are the homology S-modules of this chain complex,

$$\mathfrak{B}_i(A) := H_i(A_\bullet \otimes S).$$

- If $d = 0$, then the differentials δ_i^A are homogeneous, and so the S-modules $\mathfrak{B}_i(A)$ inherit a natural grading.
- E.g., if $E = \bigwedge V$ and $d = 0$, then $\mathfrak{B}_i(E) = 0$ for all $i \geqslant 1$.
- In general, though, $\mathfrak{B}_i(A)$ does not have a natural grading.
- Assume $\mathrm{char}(\Bbbk) = 0$ and $A^0 = \Bbbk$. An explicit finite presentation for the S-module $\mathfrak{B}(A) := \mathfrak{B}_1(A)$, was given in [Papadima–S. 2004] when $d = 0$. We generalize this presentation, as follows.

- Identify $\mathrm{MC}(A) = Z^1(A)$ with $H^1(A)$, set $r = b_1(A)$, and let $S = \mathrm{Sym}(H_1(A)) \cong \Bbbk[x_1, \dots, x_r]$ be the coordinate ring of $H^1(A)$.
- Set $E = \bigwedge H^1(A)$ and identify E^1 with $Z^1(A)$.
- Let $A^1 = E^1 \oplus U^1$ and write $A_i = (A^i)^\vee$, etc. Then $U_1 = \operatorname{im}\left(d^\vee \colon A_2 \to A_1\right)$ and $A_1 = E_1 \oplus U_1$.
- Let $\mu_A \colon A^1 \wedge A^1 \to A^2$ be the multiplication map.
- Let μ_E be the restriction of μ_A to $E^1 \wedge E^1 = E^2$, and denote its dual by $\mu_E^\vee \colon A_2 \to E_1 \wedge E_1 = E_2$.
- Let $\pi_U \colon A_1 \to U_1$ be the projection map, and set $\beta_A^\vee = (\pi_U \otimes \mathrm{id}_S) \circ (\omega_A - \mu_E \circ \omega_E)^\vee$.

Theorem

The Alexander invariant of A has presentation

$$(E_3 \oplus A_2) \otimes S \xrightarrow{\begin{pmatrix} \delta_3^E & 0 \\ \mu_E^\vee \otimes \mathrm{id}_S & d_A^\vee \otimes \mathrm{id}_S + \beta_A^\vee \end{pmatrix}} (E_2 \oplus U_1) \otimes S \longrightarrow \mathfrak{B}(A) \longrightarrow 0.$$

- Let I be the maximal ideal at 0 of the polynomial ring S.
- Its powers define a descending filtration, $\{I^n\mathfrak{B}(A)\}_{n\geqslant 0}$, on $\mathfrak{B}(A)$.
- Let $\operatorname{gr}(\mathfrak{B}(A))$ be the associated graded S-module.

PROPOSITION

For each $n \geqslant 1$, there is an isomorphism of $\Bbbk$-vector spaces,

$$\operatorname{gr}_n(\mathfrak{B}(A))^{\vee} \cong \operatorname{Tor}^{\mathcal{E}}_{n-1}(A, \Bbbk)_n,$$

where on the right side A is viewed as a graded module over the exterior algebra $\mathcal{E} = \bigwedge A^1$.

The holonomy Lie algebra of a CDGA

- Let $A = (A^\bullet, d)$ be a connected $\Bbbk$-CDGA, where $\mathrm{char}(\Bbbk) = 0$ and $\dim_\Bbbk A^1 < \infty$.
- Let $\mu^\vee : A_2 \to A_1 \wedge A_1$ be the $\Bbbk$-dual of the multiplication map $\mu\colon A^1 \wedge A^1 \to A^2$, and let $d^\vee : A_2 \to A_1$ be the dual of $d\colon A^1 \to A^2$.
- We denote by $\mathrm{Lie}(A_1)$ the free Lie algebra on A_1, and we identify $\mathrm{Lie}_1(A_1) = A_1$ and $\mathrm{Lie}_2(A_1) = A_1 \wedge A_1$.

Definition (Măcinic, Papadima, Popescu, S. 2017)

The *holonomy Lie algebra* of A is $\mathfrak{h}(A) = \mathrm{Lie}(A_1)/\langle \mathrm{im}(d^\vee + \mu^\vee)\rangle$.

- $\mathfrak{h}(A)$ is a finitely presented Lie algebra.
- In general, the ideal $\langle \mathrm{im}(d^\vee + \mu^\vee)\rangle$ is not homogeneous, and so $\mathfrak{h}(A)$ does not inherit a grading from $\mathrm{Lie}(A_1)$.
- The construction is functorial.

The infinitesimal Alexander invariant

- Let $\mathfrak{g}$ be a Lie algebra over $\Bbbk$. Set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}'' = (\mathfrak{g}')'$.
- We then have an exact sequence of Lie algebras,

$$0 \longrightarrow \mathfrak{g}'/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}'' \longrightarrow \mathfrak{g}/\mathfrak{g}' \longrightarrow 0.$$

- The adjoint representation of $\mathfrak{g}/\mathfrak{g}'$ on $\mathfrak{g}/\mathfrak{g}''$ defines an action of $S = \operatorname{Sym}(\mathfrak{g}/\mathfrak{g}')$ on $\mathfrak{g}'/\mathfrak{g}''$, given by $\bar{g} \cdot \bar{x} = \overline{[g, x]}$, for $g \in \mathfrak{g}$ and $x \in \mathfrak{g}'$.
- The *infinitesimal Alexander invariant* of $\mathfrak{g}$ is the S-module

$$\mathfrak{B}(\mathfrak{g}) := \mathfrak{g}'/\mathfrak{g}''.$$

- The construction is functorial.

Theorem

There is a natural isomorphism of S-modules, $\mathfrak{B}(A) \xrightarrow{\cong} \mathfrak{B}(\mathfrak{h}(A))$.

Holonomy Chen ranks

- For a Lie algebra $\mathfrak{g}$, we let $\operatorname{gr}(\mathfrak{g}) = \gamma_n(\mathfrak{g})/\gamma_{n+1}(\mathfrak{g})$, where $\gamma_1(\mathfrak{g}) = \mathfrak{g}$ and $\gamma_{n+1}(\mathfrak{g}) = [\mathfrak{g}, \gamma_n(\mathfrak{g})]$.
- The *holonomy Chen ranks* of A are defined as

$$\theta_n(A) := \dim_{\Bbbk} \operatorname{gr}_n(\mathfrak{h}(A)/\mathfrak{h}''(A)).$$

- Clearly, $\theta_1(A) = \dim_{\Bbbk} A^1$.
- When viewed as a module over $S = \operatorname{Sym}(H_1(A))$, the associated graded Alexander invariant, $\operatorname{gr}(\mathfrak{B}(A))$, is a finitely generated S-module, with generators in degree 0.

Proposition

The generating sequence for the holonomy Chen ranks (with a shift of 2) coincides with the Hilbert series of the graded S-module $\operatorname{gr}(\mathfrak{B}(A))$:

$$\sum_{n\geqslant 0} \theta_{n+2}(A) \cdot t^n = \operatorname{Hilb}(\operatorname{gr}(\mathfrak{B}(A)), t).$$

Resonance varieties of a CDGA

- Let $(A^{\bullet}, d)$ be a connected $\Bbbk$-CDGA with $\dim A^1 < \infty$.
- For each $a \in \mathrm{MC}(A)$, the operator $\delta_a := d + a$ is a differential on A.
- The *resonance varieties* of A:

$$\mathcal{R}^i_s(A) = \{a \in \mathrm{MC}(A) \mid \dim_{\Bbbk} H^i(A, \delta_a) \geqslant s\}.$$

- For each $i \geqslant 0$, these sets form a filtration

$$\mathrm{MC}(A) = \mathcal{R}^i_0(A) \supseteq \mathcal{R}^i_1(A) \supseteq \mathcal{R}^i_2(A) \supseteq \cdots.$$

- If $c_i := \dim_{\Bbbk} A^i < \infty$ for $i \leqslant q$ (for some $q \geqslant 1$), then $\mathcal{R}^i_s(A)$ are Zariski closed subsets of $\mathrm{MC}(A)$, for all $i \leqslant q$ and $s \geqslant 0$:

$$\mathcal{R}^i_s(A) = V\Big(I_{c_i - s+1}\big(\delta^{i-1}_A \oplus \delta^i_A\big)\Big).$$

Proposition

If $\mathrm{char}(\Bbbk) = 0$, then $\mathcal{R}^1_s(A) = V\big(\mathrm{Ann}\,\big(\bigwedge^s(\mathfrak{B}(A))\big)\big)$ for all $s \geqslant 1$, at least away from $0 \in H^1(A)$.

Poincaré duality algebras

- Let A be a connected, finite-type $\Bbbk$-CGA.
- A is a *Poincaré duality $\Bbbk$-algebra* of dimension m (m-pda) if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk,\ a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- We then have:
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*,\quad \mathrm{PD}(a)(b) = \varepsilon(ab)$ are isos.
- Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
- The *orientation class* is $\omega_A := 1^\vee$.
- We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.

- A $\Bbbk$-CDGA $(A^\bullet, d)$ is a *Poincaré duality differential graded algebra* of formal dimension m (for short, an m-PD-CDGA) if
  1. The graded algebra $A^\bullet$ is an m-PDA.
  2. $d(A^{m-1}) = 0$.
- Condition (2) can also be stated as $\varepsilon(d(u)) = 0$ for all $u \in A^{m-1}$.
- By condition (1), the algebra A is connected and $A^m \cong A^0$; thus, condition (2) is equivalent to $H^m(A) = \Bbbk$.
- If $(A^\bullet, d)$ is an m-PD-CDGA, then $H^\bullet(A)$ is an m-PDA.

LEMMA

For all $a \in A^1$ and all $0 \leqslant i \leqslant m$, we have a commuting square,

$$\begin{array}{ccc} (A^{m-i})^* & \xrightarrow{(\delta_{-a}^{m-i-1})^*} & (A^{m-i-1})^* \\ \cong \uparrow \mathrm{PD}^i & & \cong \uparrow (-1)^{i+1}\,\mathrm{PD}_{i+1} \\ A^i & \xrightarrow{\delta_a^i} & A^{i+1}\,. \end{array}$$

Resonance varieties of PD-algebras

Theorem

Let $(A^\bullet, d)$ *be an* m-PD-CDGA *over a field* $\Bbbk$. *Then,*

(1) $H^i(A, \delta_a)^* \cong H^{m-i}(A, \delta_{-a})$ *for all* $a \in \mathrm{MC}(A)$ *and* $i \geqslant 0$.

(2) The linear isomorphism $A^1 \xrightarrow{\simeq} A^1$, $a \mapsto -a$ *restricts to isomorphisms* $\mathcal{R}^i_s(A) \xrightarrow{\simeq} \mathcal{R}^{m-i}_s(A)$ *for all* $i, s \geqslant 0$.

(3) $\mathcal{R}^m_1(A) = \{0\}$.

Proposition

Let A *be a* PD_3 *algebra with* $b_1(A) = n$. *Then* $\mathcal{R}^i_k(A) = \emptyset$, *except for:*

- $\mathcal{R}^i_0(A) = A^1$ *for all* $i \geqslant 0$.
- $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$ *and* $\mathcal{R}^2_n(A) = \mathcal{R}^1_n(A) = \{0\}$.
- $\mathcal{R}^2_s(A) = \mathcal{R}^1_s(A)$ *for* $0 < s < n$.

Moreover, $\mathcal{R}^1_{2k}(A) = \mathcal{R}^1_{2k+1}(A)$ *if* n *is even, and* $\mathcal{R}^1_{2k-1}(A) = \mathcal{R}^1_{2k}(A)$ *if* n *is odd, for all* $k \geqslant 0$.

ALGEBRAIC MODELS FOR SPACES

- Given any (path-connected) space X, there is an associated Sullivan $\mathbb{Q}$-cdga, $A_{\mathrm{PL}}(X)$ such that $H^\bullet(A_{\mathrm{PL}}(X)) = H^\bullet(X, \mathbb{Q})$.
- An *algebraic (q-)model* (over field $\Bbbk$ with $\mathrm{char}(\Bbbk) = 0$) for X is a $\Bbbk$-cgda (A, d) which is (q-) equivalent to $A_{\mathrm{PL}}(X) \otimes_{\mathbb{Q}} \Bbbk$.
- A cdga A is *formal* (or just *q-formal*) if it is (q-)weakly equivalent to $(H^\bullet(A), d = 0)$.
- A CDGA A is of *finite-type* (or *q-finite*) if it is connected and each graded piece A^i (with $i \leqslant q$) is finite-dimensional.
- Examples of spaces having finite-type models include:
  - Formal spaces (such as compact Kähler manifolds, hyperplane arrangement complements, toric spaces, etc).
  - Smooth quasi-projective varieties, compact solvmanifolds, Sasakian manifolds, etc.

Associated graded Lie algebra

- The *lower central series* of a group G is defined inductively by $\gamma_1 G = G$ and $\gamma_{n+1} G = [\gamma_n G, G]$.
- This forms a filtration of G by characteristic subgroups. The LCS quotients, $\gamma_n G/\gamma_{n+1} G$, are abelian groups.
- The group commutator induces a graded Lie algebra structure on

$$\operatorname{gr}(G, \Bbbk) = \bigoplus_{n \geqslant 1} (\gamma_n G/\gamma_{n+1} G) \otimes_{\mathbb{Z}} \Bbbk.$$

- Assume G is finitely generated. Then $\operatorname{gr}(G, \Bbbk)$ is also finitely generated (in degree 1) by $\operatorname{gr}_1(G, \Bbbk) = H_1(G, \Bbbk)$.
- Let $\mathfrak{h}(G, \Bbbk) := \mathfrak{h}(H^\bullet(G, \Bbbk))$ be the *holonomy Lie algebra* of G.
- There is an epimorphism $\mathfrak{h}(G, \Bbbk) \twoheadrightarrow \operatorname{gr}(G, \Bbbk)$, which is an isomorphism precisely when $\operatorname{gr}(G, \Bbbk)$ is quadratic.

Alexander invariant

- Let $G' = [G, G]$ and $G'' = (G')'$.
- The *Alexander invariant* of G is the quotient $B(G) = G'/G''$, viewed as a $\mathbb{Z}[G_{\mathrm{ab}}]$-module via $gG' \cdot xG'' = gxg^{-1}G''$.
- If X is a connected CW-complex with $\pi_1(X) = G$, then

$$B(G) = H_1(X^{\mathrm{ab}}, \mathbb{Z}) = H_1(X, \mathbb{Z}[G_{\mathrm{ab}}]),$$

  where $X^{\mathrm{ab}} \to X$ is the universal abelian cover.
- [Massey 1980] Let $I = \ker(\varepsilon \colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z})$. Then, for all $n \geqslant 0$, $I^n B(G) = \gamma_{n+2}(G/G'')$, and thus $\mathrm{gr}_n(B) \cong \mathrm{gr}_{n+2}(G/G'')$
- In other words,

$$\mathrm{Hilb}(\mathrm{gr}(B(G) \otimes \mathbb{Q}), t) = \sum_{n \geqslant 0} \theta_{n+2}(G) t^n,$$

  where $\theta_n(G) := \operatorname{rank} \mathrm{gr}_n(G/G'')$ are the *Chen ranks* of G.

Malcev Lie algebras

- Let G be a f.g. group. The successive quotients of G by the LCS terms form a tower of finitely generated, nilpotent groups,

$$\cdots \longrightarrow G/\gamma_4 G \longrightarrow G/\gamma_3 G \longrightarrow G/\gamma_2 G = G_{\mathrm{ab}} .$$

- (Malcev 1951) It is possible to replace each nilpotent quotient N_k by $N_k \otimes \mathbb{Q}$, the nilpotent Lie group over $\mathbb{Q}$ associated to the discrete, torsion-free nilpotent group $N_k/\operatorname{tors}(N_k)$.
- The inverse limit, $\mathfrak{M}(G) = \varprojlim_n (G/\gamma_n G) \otimes \mathbb{Q}$, is a prounipotent, filtered Lie group, called the *prounipotent completion* of G.
- The pronilpotent Lie algebra $\mathfrak{m}(G) := \varprojlim_n \mathfrak{Lie}((G/\gamma_n G) \otimes \mathbb{Q})$ is called the *Malcev Lie algebra* of G (over $\Bbbk$).
- [Quillen 1968/69] $\mathfrak{m}(G) \cong \mathsf{Prim}\left(\widehat{\mathbb{Q}[G]}\right)$ and $\operatorname{gr}(\mathfrak{m}(G)) \cong \operatorname{gr}(G, \mathbb{Q})$.
- [Sullivan 1977] G is 1-formal $\Longleftrightarrow \mathfrak{m}(G)$ is quadratic.

Finiteness obstructions for groups

Theorem (Papadima–S 2019)

G admits a 1-finite 1-model if and only if $\mathfrak{m}(G)$ is the lcs completion of a finitely presented Lie algebra.

More precisely, if A is such a model (over $\Bbbk$), then $\mathfrak{m}(G) \otimes \Bbbk \cong \widehat{\mathfrak{h}(A)}$.

Example (PS19)

Let G be a metabelian group of the form $G = \pi/\pi''$, where π is a f.g. group which has a free, non-cyclic quotient. Then:

- G is not finitely presentable.
- G does not admit a 1-finite 1-model.

Alexander invariants and Chen ranks

- For a subset $\mathfrak{a} \subset \mathfrak{m}(G)$, we denote by $\overline{\mathfrak{a}}$ its closure in the topology defined by the filtration on $\mathfrak{m}(G)$.
- Let $\mathfrak{h}(G) := \mathfrak{h}(H^*(G,\mathbb{Q})) = \mathrm{Lie}(H_1(G,\mathbb{Q}))/(\mathrm{im}\,\mu^\vee)$ be the holonomy Lie algebra of G.
- Let $\mathfrak{B}(G) := \mathfrak{B}(\mathfrak{h}(G)) = \mathfrak{h}(G)'/\mathfrak{h}(G)''$ be the infinitesimal Alexander invariant of G, viewed as a graded module over $S = \mathrm{Sym}(H_1(G,\mathbb{Q}))$.

Theorem (Dimca–Papadima–S. 2009)

Let G be a finitely generated group. Then,

- *There is a filtration-preserving, $\widehat{S}$-linear isomorphism, $\widehat{B(G) \otimes \mathbb{Q}} \cong \overline{\mathfrak{m}(G)'}/\overline{\mathfrak{m}(G)''}$.*
- *If G is 1-formal, then $\widehat{B(G) \otimes \mathbb{Q}} \cong \widehat{\mathfrak{B}(G)}$.*

Theorem

Let G be a f.g. group. Suppose G admits a 1-finite 1-model, (A, d), over a field $\Bbbk$ of characteristic 0. Then

(1) $\widehat{B(G) \otimes \Bbbk} \cong \widehat{\mathfrak{B}(A)}$.

(2) $\operatorname{gr}(B(G) \otimes \Bbbk) \cong \operatorname{gr}(\mathfrak{B}(A))$.

(3) $\theta_n(G) = \theta_n(A)$ *for all* $n \geqslant 1$.

Proof.

The isomorphism in (1) is given by

$$\widehat{B(G) \otimes \Bbbk} \cong \overline{\mathfrak{m}(G)' \otimes \Bbbk} / \overline{\mathfrak{m}(G)'' \otimes \Bbbk} \cong \overline{\mathfrak{h}(A)'} / \overline{\mathfrak{h}(A)''} \cong \widehat{\mathfrak{B}(A)}.$$

Passing to associated graded gives (2). Part (3) follows from (2) and the aforementioned Massey-type equalities. □

Resonance varieties of spaces

- Let X be a connected, finite-type CW-complex.
- Let $\Bbbk$ be a field, and suppose either $\operatorname{char}(\Bbbk) \neq 2$, or $\operatorname{char}(\Bbbk) = 2$ and $H_1(X, \mathbb{Z})$ has no 2-torsion.
- Then $a^2 = 0$ for all $a \in H^1(X, \Bbbk)$.
- The *resonance varieties* of X (over $\Bbbk$) are the resonance varieties of the CDGA $A = (H^\bullet(X, \Bbbk), 0)$; that is,

$$\mathcal{R}^i_s(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid \dim_{\Bbbk} H^i(A, \delta_a) \geqslant s\},$$

where $\delta_a \colon A^i \to A^{i+1}$ is given by $\delta_a(u) = au$.

Proposition

Let M be a closed, orientable m-manifold. If $\operatorname{char}(\Bbbk) \neq 2$, then $\mathcal{R}^i_s(M; \Bbbk) = \mathcal{R}^{m-i}_s(M; \Bbbk)$ for all i, s. In particular, $\mathcal{R}^m_1(M, \Bbbk) = \{0\}$.

Bockstein resonance varieties

- Let X be a connected, finite-type CW-complex
- Let $A = H^{\bullet}(X, \mathbb{Z}_2)$, with differential given by the Bockstein operator, $\beta_2 = \mathrm{Sq}^1 \colon A^{\bullet} \to A^{\bullet+1}$.
- Since $\mathrm{Sq}^1(a) = a^2$ for all $a \in A^1$, the Maurer–Cartan set for the CDGA (A, β_2) is then $\mathrm{MC}(A) = A^1$.
- The *Aomoto–Bockstein complex* of A with respect to $a \in A^1$:

$$(A, \delta_a)\colon\ A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} \cdots \xrightarrow{\delta_a} A^i \xrightarrow{\delta_a} A^{i+1} \xrightarrow{\delta_a} \cdots,$$

  where $\delta_a(u) = au + \beta_2(u)$.
- Pick basis $\{e_1, \dots, e_n\}$ for $A^1 = H^1(X, \mathbb{Z}_2)$, let $\{x_1, \dots, x_n\}$ be dual basis for $A_1 = H_1(X, \mathbb{Z}_2)$, and identify $\mathrm{Sym}(A_1) \cong \mathbb{Z}_2[x_1, \dots, x_n]$.
- The coordinate ring of A^1 is then

$$S = \mathbb{Z}_2[x_1, \dots, x_n]/(x_1^2 + x_1, \dots, x_n^2 + x_n).$$

  This is the ring of (Boolean) functions on $\mathbb{Z}_2^n$.

- The *universal Aomoto–Bockstein complex* of X is the cochain complex of free S-modules,

$$(A \otimes_{\mathbb{Z}_2} S, \delta)\colon\ A^0 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^0} A^1 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^1} A^2 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^2} \cdots,$$

  where $\delta^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j + \beta_2(u) \otimes 1$ for $u \in A^i$.

- Example: If $X = \mathbb{RP}^\infty$, then $H^\bullet(X, \mathbb{Z}_2) = \mathbb{Z}_2[a]$, where $|a| = 1$, and $\beta_2(a^i) = a^{i+1}$ if i is odd and $\beta_2(a^i) = 0$ if i is even. Setting $S = \mathbb{Z}_2[x]/(x^2 + x)$, we get the (exact) cochain complex

$$S \xrightarrow{x} S \xrightarrow{x+1} S \xrightarrow{x} S \longrightarrow \cdots.$$

- The *Bockstein resonance varieties* of X are the resonance varieties of the CDGA $A = (H^\bullet(X, \mathbb{Z}_2), \beta_2)$; that is,

$$\widetilde{\mathcal{R}}^q_s(X, \mathbb{Z}_2) = \{a \in H^1(X, \mathbb{Z}_2) \mid \dim_{\mathbb{Z}_2} H^q(A, \delta_a) \geqslant s\},$$

  where $\delta_a\colon A^q \to A^{q+1}$ is given by $\delta_a(u) = au + \beta_2(u)$.

- More generally, if $\mathrm{char}(\Bbbk) = 2$, then $\widetilde{\mathcal{R}}^q_s(X, \Bbbk) = \widetilde{\mathcal{R}}^q_s(X, \mathbb{Z}_2) \times_{\mathbb{Z}_2} \Bbbk$.

- If $H_1(X,\mathbb{Z})$ has no 2-torsion, then $\mathcal{R}^1_s(X,\mathbb{Z}_2) = \widetilde{\mathcal{R}}^1_s(X,\mathbb{Z}_2)$, $\forall s$.
- $\mathcal{R}^q_s(X,\mathbb{Z}_2) \neq \widetilde{\mathcal{R}}^q_s(X,\mathbb{Z}_2)$ for $q > 1$ (neither inclusion needs to hold).

THEOREM

Let M be a closed m-manifold. The following are equivalent:

(1) M *is orientable*

(2) $\beta_2 \colon H^{m-1}(M,\mathbb{Z}_2) \to H^m(M,\mathbb{Z}_2)$ *is zero.*

(3) $(H^{\bullet}(M,\mathbb{Z}_2), \beta_2)$ *is an* m-PD-CDGA.

(4) $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \{0\}$.

PROPOSITION

Let M be a closed, orientable m-manifold, and assume $\mathrm{char}(\Bbbk) = 2$. *Then $\widetilde{\mathcal{R}}^i_s(M;\Bbbk) = \widetilde{\mathcal{R}}^{m-i}_s(M;\Bbbk)$ for all i, s. In particular, $\widetilde{\mathcal{R}}^m_1(M,\Bbbk) = \{0\}$.*

PROPOSITION

Let M be a closed, non-orientable m-manifold such that $H_1(M,\mathbb{Z})$ has no 2-torsion. Then $\mathcal{R}^m_1(M,\mathbb{Z}_2) = \{0\}$ whereas $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2) = \mathbb{Z}_2$.

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