Sigma-invariants and tropical varieties

Alex Suciu

Northeastern University, Boston

Conference on Finiteness Conditions in Topology and Algebra

Queen's University Belfast

September 1, 2015

1. Finiteness properties of abelian covers
   - The Bieri–Neumann–Strebel–Renz invariants
   - The Dwyer–Fried invariants
2. Characteristic varieties and finiteness properties
   - Jump loci and exponential tangent cones
   - Resonance varieties
   - Bounding the Σ-invariants
   - A formula and a bound for the Ω-invariants
   - Comparing the Σ- and Ω-bounds
3. Tropical bounds
   - Tropical geometry
   - Tropicalizing the characteristic varieties
4. Applications
   - Toric complexes and right-angled Artin groups
   - Hyperplane arrangements
   - Kähler manifolds

THE BIERI–NEUMANN–STREBEL–RENZ INVARIANTS

- Let π be a finitely generated group, $n = b_1(\pi) > 0$. Let $S(\pi)$ be the unit sphere in $\mathrm{Hom}(\pi, \mathbb{R}) = \mathbb{R}^n$.
- The BNSR-invariants of π form a descending chain of open subsets, $S(\pi) \supseteq \Sigma^1(\pi, \mathbb{Z}) \supseteq \Sigma^2(\pi, \mathbb{Z}) \supseteq \cdots$.
- $\Sigma^k(\pi, \mathbb{Z})$ consists of all $\chi \in S(\pi)$ for which the monoid $\pi_\chi = \{g \in \pi \mid \chi(g) \geq 0\}$ is of type FP_k, i.e., there is a projective $\mathbb{Z}\pi$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft \pi$ for which π/N is free abelian:

$$N \text{ is of type } \mathrm{FP}_k \iff S(\pi, N) \subseteq \Sigma^k(\pi, \mathbb{Z})$$

  where $S(\pi, N) = \{\chi \in S(\pi) \mid \chi(N) = 0\}$.
- In particular: $\ker(\chi \colon \pi \twoheadrightarrow \mathbb{Z})$ is f.g. $\iff \{\pm\chi\} \subseteq \Sigma^1(\pi, \mathbb{Z})$.

- More generally, let X be a connected CW-complex with finite k-skeleton, for some $k \geq 1$.
- Let $\pi = \pi_1(X, x_0)$. For each $\chi \in S(X) := S(\pi)$, set

$$\widehat{\mathbb{Z}\pi}_\chi = \{\lambda \in \mathbb{Z}^\pi \mid \{g \in \operatorname{supp} \lambda \mid \chi(g) < c\} \text{ is finite}, \forall c \in \mathbb{R}\}$$

be the Novikov–Sikorav completion of $\mathbb{Z}\pi$.

- Following Farber, Geoghegan, and Schütz (2010), define

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in S(X) \mid H_i(X, \widehat{\mathbb{Z}\pi}_{-\chi}) = 0, \ \forall\, i \leq q\}.$$

- (Bieri) If π is FP_k, then $\Sigma^q(\pi, \mathbb{Z}) = \Sigma^q(K(\pi, 1), \mathbb{Z})$, $\forall q \leq k$.
- The sphere $S(\pi)$ parametrizes all regular, free abelian covers of X. The Σ-invariants of X keep track of the geometric finiteness properties of these covers.

THE DWYER–FRIED INVARIANTS

- Now fix the rank r of the deck-transformation group.
- Regular $\mathbb{Z}^r$-covers of X are classified by epimorphisms $\nu\colon \pi \twoheadrightarrow \mathbb{Z}^r$.
- Such covers are parameterized by the Grassmannian $\mathrm{Gr}_r(\mathbb{Q}^n)$, where $n = b_1(X)$, via the correspondence

$$\{\text{regular } \mathbb{Z}^r\text{-covers of } X\} \longleftrightarrow \{r\text{-planes in } H^1(X,\mathbb{Q})\}$$
$$X^\nu \to X \longleftrightarrow P_\nu := \mathrm{im}(\nu^*\colon \mathbb{Q}^r \to H^1(X,\mathbb{Q}))$$

- The *Dwyer–Fried invariants* of X are the subsets

$$\Omega^i_r(X) = \{P_\nu \in \mathrm{Gr}_r(\mathbb{Q}^n) \mid b_j(X^\nu) < \infty \text{ for } j \le i\}.$$

- For each $r > 0$, we get a descending filtration,

$$\mathrm{Gr}_r(\mathbb{Q}^n) = \Omega^0_r(X) \supseteq \Omega^1_r(X) \supseteq \Omega^2_r(X) \supseteq \cdots .$$

CHARACTERISTIC VARIETIES

- Let $\widehat{\pi} = \operatorname{Hom}(\pi, \mathbb{C}^*) = H^1(X, \mathbb{C}^*)$ be the character group of $\pi = \pi_1(X)$.
- The *characteristic varieties* of X are the sets

$$\mathcal{V}^i(X) = \{\rho \in \widehat{\pi} \mid H_i(X, \mathbb{C}_\rho) \neq 0\}.$$

- If X has finite k-skeleton, then $\mathcal{V}^i(X)$ is a Zariski closed subset of the algebraic group $\widehat{\pi}$, for each $i \leq k$.
- Let $X^{\mathrm{ab}} \to X$ be the maximal abelian cover. View $H_*(X^{\mathrm{ab}}, \mathbb{C})$ as a module over $\mathbb{C}[\pi_{\mathrm{ab}}]$. Then

$$\bigcup_{i \leq j} \mathcal{V}^i(X) = \bigcup_{i \leq j} V\left(\operatorname{ann}\left(H_i(X^{\mathrm{ab}}, \mathbb{C})\right)\right).$$

- Moreover, $\mathcal{V}^1(X) \cap \widehat{\pi}^0 = \{1\} \cup V(\Delta_\pi)$, where Δ_π is the Alexander polynomial of π.

Propagation of jump loci

- Bieri–Eckmann (1973): X is a *duality space* of dimension d if $H^i(X,\mathbb{Z}\pi)=0$ for $i\neq d$, while $H^d(X,\mathbb{Z}\pi)\neq 0$ and torsion-free.
- We say X is an *abelian duality space* of dimension d if $H^i(X,\mathbb{Z}\pi_{\mathrm{ab}})=0$ for $i\neq d$, while $H^d(X,\mathbb{Z}\pi_{\mathrm{ab}})\neq 0$ and torsion-free.
- Let $B=H^d(X,\mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X,A)\cong H_{n-i}(X,B\otimes A)$.

Theorem (Denham–S.–Yuzvinsky 2015)

Let X be an abelian duality space of dimension d. If $\rho\colon \pi_1(X)\to\mathbb{C}^*$ satisfies $H^i(X,\mathbb{C}_\rho)\neq 0$, then $H^j(X,\mathbb{C}_\rho)\neq 0$, for all $i\leq j\leq d$. Consequently,

- *The characteristic varieties propagate, i.e., $\mathcal{V}^1(X)\subseteq\cdots\subseteq\mathcal{V}^d(X)$.*
- $\dim H^1(X,\mathbb{C})\geq d-1$.
- *If $d\geq 2$, then $H^i(X,\mathbb{C})\neq 0$, for all $0\leq i\leq d$.*

Exponential tangent cones

- Let $\exp\colon H^1(X,\mathbb{C}) \to H^1(X,\mathbb{C}^*)$ be the coefficient homomorphism induced by $\mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$.
- Given a Zariski closed subset $W \subset H^1(X,\mathbb{C}^*)$, define the *exponential tangent cone* of W at 1 as

$$\tau_1(W) = \{z \in H^1(X,\mathbb{C}) \mid \exp(\lambda z) \in W,\ \forall \lambda \in \mathbb{C}\}.$$

- $\tau_1(W)$ is a finite union of rationally defined linear subspaces.
- $\tau_1(W)$ is non-empty iff $1 \in W$.
- For instance, if $T \cong (\mathbb{C}^*)^r$ is an algebraic subtorus, then $\tau_1(T) = T_1(T) \cong \mathbb{C}^r$.
- Set $\tau_1^{\Bbbk}(W) = \tau_1(W) \cap H^1(X,\Bbbk)$, for a subfield $\Bbbk \subset \mathbb{C}$.

RESONANCE VARIETIES

- Let $A = H^*(X, \mathbb{C})$. For each $a \in A^1$, we have that $a^2 = 0$. Thus, there is a cochain complex $(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$.
- The *resonance varieties* of X are the homogeneous algebraic sets

$$\mathcal{R}^i(X) = \{a \in A^1 \mid H^i(A, a) \neq 0\}.$$

THEOREM (LIBGOBER 2002, DIMCA–PAPADIMA–S. 2009)

$$\tau_1(\mathcal{V}^i(X)) \subseteq \mathrm{TC}_1(\mathcal{V}^i(X)) \subseteq \mathcal{R}^i(X).$$

THEOREM (DPS-2009, DP-2014)

Suppose X is a q-formal space. Then, for all $i \leq q$,

$$\tau_1(\mathcal{V}^i(X)) = \mathrm{TC}_1(\mathcal{V}^i(X)) = \mathcal{R}^i(X).$$

BOUNDING THE Σ-INVARIANTS

- Let $\chi \in S(X)$, and set $\Gamma = \operatorname{im}(\chi)$; then $\Gamma \cong \mathbb{Z}^r$, for some $r \geq 1$.
- A Laurent polynomial $p = \sum_\gamma n_\gamma \gamma \in \mathbb{Z}\Gamma$ is χ-*monic* if the greatest element in $\chi(\operatorname{supp}(p))$ is 0, and $n_0 = 1$.
- Let $\mathcal{R}\Gamma_\chi$ be the Novikov ring, i.e., the localization of $\mathbb{Z}\Gamma$ at the multiplicative subset of all χ-monic polynomials (it's a PID).
- Let $b_i(X,\chi) = \operatorname{rank}_{\mathcal{R}\Gamma_\chi} H_i(X, \mathcal{R}\Gamma_\chi)$ be the Novikov–Betti numbers.

THEOREM (PAPADIMA–S. 2010)

- $-\chi \in \Sigma^k(X,\mathbb{Z}) \implies b_i(X,\chi) = 0, \forall i \leq k.$
- $\chi \notin \tau_1^{\mathbb{R}}(\bigcup_{q\leq i} \mathcal{V}^q(X)) \iff b_i(X,\chi) = 0, \forall i \leq k.$

Hence, $\Sigma^i(X,\mathbb{Z}) \subseteq S(X) \setminus S(\tau_1^{\mathbb{R}}(\bigcup_{q\leq i} \mathcal{V}^q(X))).$

Thus, $\Sigma^i(X,\mathbb{Z})$ is contained in the complement of a finite union of rationally defined great subspheres.

A FORMULA AND A BOUND FOR THE Ω-INVARIANTS

THEOREM (DWYER–FRIED 1987, PAPADIMA–S. 2010)

For an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \mathbb{Z}^r$*, the following are equivalent:*

- *The vector space* $\bigoplus_{i=0}^{k} H_i(X^\nu, \mathbb{C})$ *is finite-dimensional.*
- *The algebraic torus* $\mathbb{T}_\nu = \operatorname{im}\left(\hat{\nu}\colon \widehat{\mathbb{Z}^r} \hookrightarrow \widehat{\pi_1(X)}\right)$ *intersects the variety* $\mathcal{W}^k(X) = \bigcup_{i \le k} \mathcal{V}^i(X)$ *in only finitely many points.*

Note that $\exp(P_\nu \otimes \mathbb{C}) = \mathbb{T}_\nu$. Thus:

COROLLARY

$$\Omega_r^i(X) = \left\{P \in \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \mid \dim\left(\exp(P \otimes \mathbb{C}) \cap \mathcal{W}^i(X)\right) = 0\right\}$$

COROLLARY

- *If* $\mathcal{W}^i(X)$ *is finite, then* $\Omega_r^i(X) = \operatorname{Gr}_r(\mathbb{Q}^n)$*, where* $n = b_1(X)$*.*
- *If* $\mathcal{W}^i(X)$ *is infinite, then* $\Omega_n^q(X) = \emptyset$*, for all* $q \ge i$*.*

- Let V be a homogeneous variety in $\Bbbk^n$. The set $\sigma_r(V) = \{P \in \operatorname{Gr}_r(\Bbbk^n) \mid P \cap V \neq \{0\}\}$ is Zariski closed.
- If $L \subset \Bbbk^n$ is a linear subspace, $\sigma_r(L)$ is the *special Schubert variety* defined by L. If $\operatorname{codim} L = d$, then $\operatorname{codim} \sigma_r(L) = d - r + 1$.

THEOREM

$$\Omega_r^i(X) \subseteq \operatorname{Gr}_r(H^1(X, \mathbb{Q})) \setminus \sigma_r\big(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X))\big)$$

- Thus, each set $\Omega_r^i(X)$ is contained in the complement of a finite union of special Schubert varieties.
- If $r = 1$, the inclusion always holds as an equality. In general, though, the inclusion is strict.

EXAMPLE

Let $\pi = \langle x_1, x_2, x_3 \mid [x_1^2, x_2],\ [x_1, x_3],\ x_1[x_2, x_3]x_1^{-1}[x_2, x_3]\rangle$. Then

$$\mathcal{V}^1(\pi) = \{1\} \cup \big\{t \in (\mathbb{C}^*)^3 \mid t_1 = -1\big\}.$$

Thus, $\Omega_2^1(\pi)$ is a single point in $\operatorname{Gr}_2(H^1(G, \mathbb{Q})) = \mathbb{QP}^2$, hence *not* open.

COMPARING THE Σ- AND Ω-BOUNDS

THEOREM

Suppose that $\Sigma^i(X,\mathbb{Z}) = S(X) \setminus S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X)))$.
Then $\Omega_r^i(X) = \mathrm{Gr}_r(H^1(X,\mathbb{Q})) \setminus \sigma_r(\tau_1^{\mathbb{Q}}(\mathcal{W}^i(X)))$, *for all* $r \geq 1$.

COROLLARY

Suppose there is an integer $r \geq 2$ *such that* $\Omega_r^i(X)$ *is* not *Zariski open.*
Then $\Sigma^i(X,\mathbb{Z}) \neq S(\tau_1^{\mathbb{R}}(\mathcal{W}^i(X)))^{\complement}$.

In general, the implication from the theorem cannot be reversed.

EXAMPLE

Let $\pi = \mathrm{BS}(1,2) = \langle x_1, x_2 \mid x_1 x_2 x_1^{-1} = x_2^2 \rangle$. Then $\mathcal{V}^1(\pi) = \{1,2\} \subset \mathbb{C}^*$.
Thus, $\Omega_1^1(\pi) = \{\mathrm{pt}\}$, and so $\Omega_1^1(\pi) = \sigma_1(\tau_1^{\mathbb{Q}}(\mathcal{V}^1(\pi)))^{\complement}$.
On the other hand, $\Sigma^1(\pi) = \{-1\}$, whereas $S(\tau_1^{\mathbb{Q}}(\mathcal{V}^1(\pi)))^{\complement} = \{\pm 1\}$.

TROPICAL GEOMETRY

- Let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$.
- A non-zero element of $\mathbb{K}$ has the form $c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $c_i \in \mathbb{C}^*$ and $a_1 < a_2 < \cdots$ are rational numbers with a common denominator.
- The (algebraically closed) field $\mathbb{K}$ admits a discrete valuation $v \colon \mathbb{K}^* \to \mathbb{Q}$, given by $v(c(t)) = a_1$.
- Let $v \colon (\mathbb{K}^*)^n \to \mathbb{Q}^n \subset \mathbb{R}^n$ be the n-fold product of the valuation.
- The *tropicalization* of a variety $W \subset (\mathbb{K}^*)^n$, denoted $\text{Trop}(W)$, is the closure of the set $v(W)$ in $\mathbb{R}^n$.
- This is a rational polyhedral complex in $\mathbb{R}^n$. For instance, if W is a curve, then $\text{Trop}(W)$ is a graph with rational edge directions.

- If T be an algebraic subtorus of $(\mathbb{K}^*)^n$, then $\mathrm{Trop}(T)$ is the linear subspace $\mathrm{Hom}(\mathbb{K}^*, T) \otimes \mathbb{R} \subset \mathrm{Hom}(\mathbb{K}^*, (\mathbb{K}^*)^n) \otimes \mathbb{R} = \mathbb{R}^n$.
- Moreover, if $z \in (\mathbb{K}^*)^n$, then $\mathrm{Trop}(z \cdot T) = \mathrm{Trop}(T) + v(z)$.
- For a variety $W \subset (\mathbb{C}^*)^n$, we may define its tropicalization by setting $\mathrm{Trop}(W) = \mathrm{Trop}(W \times_{\mathbb{C}} \mathbb{K})$.
- In this case, the tropicalization is a polyhedral fan in $\mathbb{R}^n$.
- For instance, if $W = V(f)$ is a hypersurface, defined by a Laurent polynomial $f \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, then $\mathrm{Trop}(W)$ is the positive codimension- skeleton of the inner normal fan to the Newton polytope of f.

LEMMA

Let $W \subset (\mathbb{C}^*)^n$ be an algebraic variety. Then $\tau_1^{\mathbb{R}}(W) \subseteq \mathrm{Trop}(W)$.

TROPICALIZING THE CHARACTERISTIC VARIETIES

- Let X be a connected CW-complex w/ finite k-skeleton, $n = b_1(X)$.
- For each $q \leq k$, let $\mathcal{W}^q(X) = \bigcup_{i \leq q} \mathcal{V}^i(X) \cap H^1(X, \mathbb{C}^*)^0 \subset (\mathbb{C}^*)^n$.
- Let $\operatorname{Trop}(\mathcal{W}^q(X)) \subset \mathbb{R}^n$ be its tropicalization.

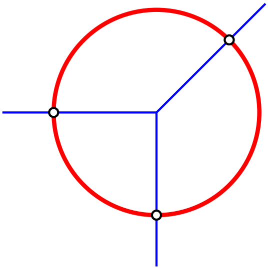
THEOREM

$$\Sigma^q(X, \mathbb{Z}) \subseteq S(X) \setminus S(\operatorname{Trop}(\mathcal{W}^q(X))).$$

COROLLARY

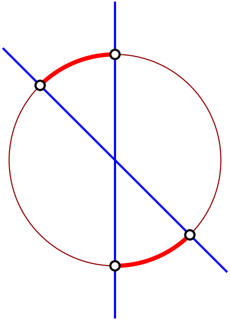
Let π be a finitely generated group, and let Δ_π be its Alexander polynomial. Then:

$$\Sigma^1(\pi) \subseteq S(\pi) \setminus S(\operatorname{Trop}(V(\Delta_\pi))).$$



EXAMPLE

- Let $\pi = \langle a, b \mid a^{-1}b^2ab^{-2} = aba^{-1}b^{-1}\rangle$.
- By Brown's algorithm, $\Sigma^1(\pi, \mathbb{Z}) = S^1 \setminus \{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -1), (-1, 0)\}$.
- On the other hand, $\Delta_\pi = 1 + b - a$.
- Thus, $\Sigma^1(G) = S(\operatorname{Trop}(V(\Delta_\pi)))^\complement$, although $\tau_1(\mathcal{V}^1(\pi)) = \{0\}$.



EXAMPLE

- Let $\pi = \langle a, b \mid a^2ba^{-1}ba^2ba^{-1}b^{-3}a^{-1}ba^2ba^{-1}bab^{-1}a^{-2}b^{-1}\cdot ab^{-1}a^{-2}b^{-1}ab^3ab^{-1}a^{-2}b^{-1}ab^{-1}a^{-1}b\rangle$ (Dunfield's link group).
- Then $\Delta_\pi = (a-1)(ab-1)$, and so $S(\mathrm{Trop}(V(\Delta_\pi)))$ consists of 4 points.
- Yet $\Sigma^1(\pi, \mathbb{Z})$ consists of two open arcs joining those two pairs of points. Thus, the tropical bound is strict in this case.

TORIC COMPLEXES AND RAAGS

- Let L be a d-dimensional simplicial complex on vertex set V with $|\mathsf{V}| = n$.
- The *toric complex* T_L is the subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L.
- T_L is a connected CW-complex, of dimension $d+1$. Moreover, T_L is formal.
- $\pi_\Gamma := \pi_1(T_L)$ is the *right-angled Artin group* associated to the graph $\Gamma = L^{(1)}$.
- $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.
- $H^*(T_L, \mathbb{Z})$ is the exterior Stanley-Reisner ring of L, with generators the duals v^*, and relations the monomials corresponding to the missing simplices of L.

L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\mathrm{lk}(\sigma)$ is concentrated in degree $d - |\sigma|$ and is torsion-free.

THEOREM (N. BRADY–MEIER 2001, JENSEN–MEIER 2005)

A right-angled Artin group π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

THEOREM (DSY 2015)

A toric complex T_L is an abelian duality space if and only if L is Cohen-Macaulay, in which case the characteristic varieties of T_L propagate.

- Identify $\widehat{\pi_\Gamma} = H^1(T_L, \mathbb{C}^*)$ with $(\mathbb{C}^*)^{\mathsf{V}} = (\mathbb{C}^*)^n$.
- Each subset $\mathsf{W} \subseteq \mathsf{V}$ yields an algebraic subtorus $(\mathbb{C}^*)^{\mathsf{W}} \subset (\mathbb{C}^*)^{\mathsf{V}}$.

THEOREM (PAPADIMA–S. 2009)

$$\mathcal{V}^i(T_L) = \bigcup_{\mathsf{W}} (\mathbb{C}^*)^{\mathsf{W}} \quad \text{and} \quad \mathcal{R}^i(T_L) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}},$$

where the union is taken over all $\mathsf{W} \subseteq \mathsf{V}$ *for which there is a simplex* $\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}$ *and an index* $j \leq i$ *such that* $\widetilde{H}_{j-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \mathbb{C}) \neq 0$.

COROLLARY

$$\Omega^i_r(T_L) = \mathrm{Gr}_r(\mathbb{Q}^{\mathsf{V}}) \setminus \sigma_r(\mathcal{R}^i(T_L, \mathbb{Q})).$$

Using results of Meier–Meinert–VanWyk & Bux–Gonzalez, we get:

COROLLARY

$$\Sigma^i(\pi_\Gamma, \mathbb{R}) = S(\mathcal{R}^i(\pi_\Gamma, \mathbb{R}))^{\mathsf{c}}.$$

Hyperplane arrangements

- Let $\mathcal{A} = \{H_1, \dots H_n\}$ be an (essential, central) arrangement of hyperplanes in $\mathbb{C}^d$.
- Its complement, $M(\mathcal{A}) \subset (\mathbb{C}^*)^n$, is a Stein manifold, and thus has the homotopy type of d-dimensional CW-complex.
- $\operatorname{Trop}(M(\mathcal{A}))$ is the 'Bergman fan' of the underlying matroid of $\mathcal{A}$.

Theorem (Davis–Januszkiewicz–Okun (2011), DSY (2015))

Suppose $A = \mathbb{Z}[\pi]$ or $A = \mathbb{Z}[\pi_{ab}]$. Then $H^p(M(\mathcal{A}), A) = 0$ for all $p \neq d$, and $H^d(M(\mathcal{A}), A)$ is a free abelian group.

Corollary

- *$M(\mathcal{A})$ is a duality and an abelian duality space of dimension d.*
- *The characteristic varieties of $M(\mathcal{A})$ propagate.*

- The cohomology ring $H^*(M(\mathcal{A}),\mathbb{Z})$ is the Orlik–Solomon algebra of the underlying matroid. Moreover, $M(\mathcal{A})$ is formal.
- Work of Arapura, Falk, D.Cohen–A.S., Libgober–Yuzvinsky, and Falk–Yuzvinsky completely describes the resonance varieties $\mathcal{R}^1(\mathcal{A}) = \mathcal{R}^1(M(\mathcal{A}),\mathbb{C})$:
  - $\mathcal{R}^1(\mathcal{A})$ is a union of linear subspaces in $H^1(M(\mathcal{A}),\mathbb{C}) \cong \mathbb{C}^{|\mathcal{A}|}$.
  - Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
  - Each *k-multinet* on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of $\mathcal{R}^1(\mathcal{A})$ of dimension $k-1$. Moreover, all components of $\mathcal{R}^1(\mathcal{A})$ arise in this way.

QUESTION (S., AT OBERWOLFACH MINIWORKSHOP 2007)

Given an arrangement $\mathcal{A}$, do we have

$$\Sigma^1(M(\mathcal{A}), \mathbb{Z}) = S(\mathcal{R}^1(M(\mathcal{A}), \mathbb{R}))^{\complement}? \qquad (\star)$$

EXAMPLE (KOBAN–MCCAMMOND–MEIER 2013)

- Let $\mathcal{A}$ be the braid arrangement in $\mathbb{C}^n$, defined by $\prod_{1\leq i<j\leq n}(z_i - z_j) = 0$. Then $M(\mathcal{A}) = \mathrm{Conf}(n, \mathbb{C}) \simeq K(P_n, 1)$.
- Answer to $(\star)$ is yes: $\Sigma^1(M(\mathcal{A}), \mathbb{Z})$ is the complement of the union of $\binom{n}{3} + \binom{n}{4}$ planes in $\mathbb{C}^{\binom{n}{2}}$, intersected with the unit sphere.

EXAMPLE (S.)

- Let $\mathcal{A}$ be the "deleted B_3" arrangement, defined by $z_1 z_2 (z_1^2 - z_2^2)(z_1^2 - z_2^2)(z_2^2 - z_3^2) = 0$.
- $\mathcal{R}^1(M(\mathcal{A}), \mathbb{R})) \subsetneq \mathrm{Trop}(\mathcal{V}^1(M(\mathcal{A}))$, and so the answer to $(\star)$ is no.

KÄHLER MANIFOLDS

THEOREM (DELZANT 2010, PAPADIMA–S. 2010)

Let M be a compact Kähler manifold with $b_1(M) > 0$. Then

$$\Sigma^1(M,\mathbb{Z}) = S(\mathcal{R}^1(M,\mathbb{R}))^{\mathsf{c}}$$

if and only if there is no pencil $f\colon M \to E$ onto an elliptic curve E such that f has multiple fibers.

THEOREM (S. 2013)

- *If M admits an orbifold fibration with base genus $g \geq 2$, then $\Omega^1_r(M) = \emptyset$, for all $r > b_1(M) - 2g$.*
- *Otherwise, $\Omega^1_r(M) = \mathrm{Gr}_r(H^1(M,\mathbb{Q}))$, for all $r \geq 1$.*
- *Suppose M admits an orbifold fibration with multiple fibers and base genus $g = 1$. Then $\Omega^1_2(M)$ is* not *an open subset of $\mathrm{Gr}_2(H^1(M,\mathbb{Q}))$.*