The rational cohomology of real quasi-toric manifolds

Alex Suciu

Northeastern University

Joint work with Alvise Trevisan (VU Amsterdam)

Toric Methods in Homotopy Theory
Queen's University Belfast
July 20, 2011

Quasi-toric manifolds and small covers

- Let P be an n-dimensional convex polytope; facets $F_1, \dots, F_m$.
- Assume P is *simple* (each vertex is the intersection of n facets).
- Then P determines a dual simplicial complex, $K = K_{\partial P}$, of dimension $n - 1$:
  - Vertex set $[m] = \{1, \dots, m\}$.
  - Add a simplex $\sigma = (i_1, \dots, i_k)$ whenever $F_{i_1}, \dots, F_{i_k}$ intersect.

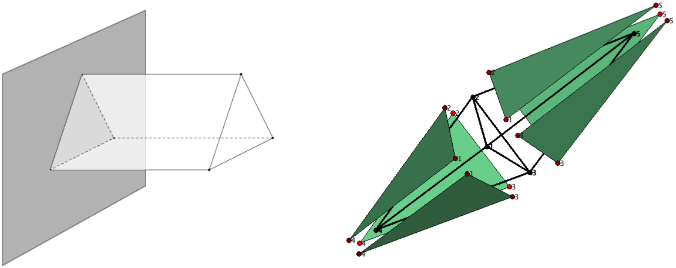


Figure: A prism P and its dual simplicial complex K

- Let χ be an n-by-m matrix with coefficients in $G = \mathbb{Z}$ or $\mathbb{Z}_2$.
- χ is *characteristic* for P if, for each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$, the n-by-n minor given by the columns $i_1, \dots, i_n$ of χ is unimodular.
- Let $\mathbb{T} = S^1$ if $G = \mathbb{Z}$, and $\mathbb{T} = S^0 = \{\pm 1\}$ if $G = \mathbb{Z}_2$.
- Given $q \in P$, let $F(q) = F_{j_1} \cap \cdots \cap F_{j_k}$ be the maximal face so that $q \in F(q)^\circ$.
- The map χ associates to $F(q)$ a subtorus $\mathbb{T}_{F(q)} \cong \mathbb{T}^k$ inside $\mathbb{T}^n$.
- To the pair (P, χ), Davis and Januszkiewicz associate the *quasi-toric manifold*

$$\mathbb{T}^n \times P/\sim, \text{ where } (t,p) \sim (u,q) \text{ if } p = q \text{ and } t \cdot u^{-1} \in \mathbb{T}_{F(q)}.$$

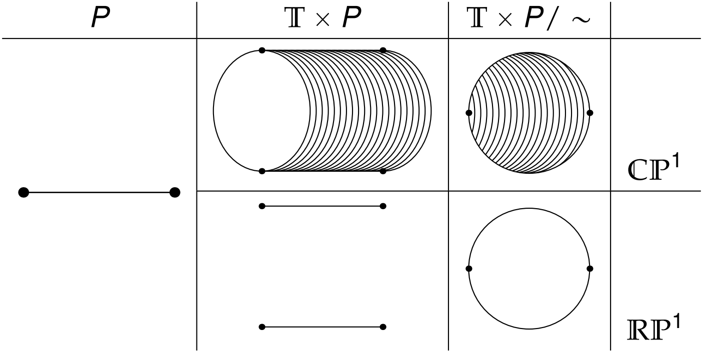
- For $G = \mathbb{Z}$, this is a *complex* q-tm, denoted $M_P(\chi)$
  - a closed, orientable manifold of dimension $2n$.
- For $G = \mathbb{Z}_2$, this is a *real* q-tm (or, *small cover*), denoted $N_P(\chi)$
  - a closed, not necessarily orientable manifold of dimension n.

Example

Let $P = \Delta^n$ be the n-simplex, and χ the $n \times (n+1)$ matrix $\begin{pmatrix} 1 & \cdots & 0 & 1 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}$.

Then

$$M_P(\chi) = \mathbb{CP}^n \quad \text{and} \quad N_P(\chi) = \mathbb{RP}^n.$$



- More generally, if X is a smooth, projective toric variety, then $X(\mathbb{C}) = M_P(\chi)$ and $X(\mathbb{R}) = N_P(\chi \bmod 2\mathbb{Z})$.
- But the converse does not hold:
  - $M = \mathbb{CP}^2 \sharp \mathbb{CP}^2$ is a quasi-toric manifold over the square, but it does not admit any complex structure. Thus, $M \not\cong X(\mathbb{C})$.
  - If P is a 3-dim polytope with no triangular or quadrangular faces, then, by a theorem of Andreev, $N_P(\chi)$ is a hyperbolic 3-manifold. (Characteristic χ exist for $P =$ dodecahedron, by work of Garrison and Scott.) Then, by a theorem of Delaunay, $N_P(\chi) \not\cong X(\mathbb{R})$.
- Davis and Januszkiewicz found presentations for the cohomology rings $H^*(M_P(\chi), \mathbb{Z})$ and $H^*(N_P(\chi), \mathbb{Z}_2)$, similar to the ones given by Danilov and Jurkiewicz for toric varieties. In particular,

$$\dim_{\mathbb{Q}} H_{2i}(M_P(\chi), \mathbb{Q}) = \dim_{\mathbb{Z}_2} H_i(N_P(\chi), \mathbb{Z}_2) = h_i(P),$$

  where $(h_0(P), \dots, h_n(P))$ is the h-vector of P, depending only on the number of i-faces of P ($0 \leqslant i \leqslant n$).

- Our goal is to compute $H^*(N_P(\chi), \mathbb{Q})$, both additively and multiplicatively.
- The Betti numbers of $N_P(\chi)$ no longer depend just on the h-vector of P, but also on the characteristic matrix χ.

Example

Let P be the square (with $n = 2$, $m = 4$). There are precisely two small covers over P:

- The torus $T^2 = N_P(\chi)$, with $\chi = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$.
- The Klein bottle $K\ell = N_P(\chi')$, with $\chi' = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right)$.

Then $b_1(T^2) = 2$, yet $b_1(K\ell) = 1$.

- Key ingredient in our approach: use finite covers involving (up to homotopy) certain generalized moment-angle complexes:

$$\mathbb{Z}_2^{m-n} \longrightarrow \mathcal{Z}_K(S^1, S^0) \longrightarrow N_P(\chi) \; ,$$

$$\mathbb{Z}_2^{n} \longrightarrow N_P(\chi) \longrightarrow \mathcal{Z}_K(\mathbb{RP}^\infty, *) \; .$$

Generalized moment-angle complexes

- Let (X, A) be a pair of topological spaces, and K a simplicial complex on vertex set $[m]$.
- The corresponding *generalized moment-angle complex* is

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^{\sigma} \subset X^{\times m}$$

  where $(X, A)^{\sigma} = \{x \in X^{\times m} \mid x_i \in A \text{ if } i \notin \sigma\}$.
- Construction interpolates between $A^{\times m}$ and $X^{\times m}$.
- Homotopy invariance:
  $(X, A) \simeq (X', A') \implies \mathcal{Z}_K(X, A) \simeq \mathcal{Z}_K(X', A')$.
- Converts simplicial joins to direct products:
  $\mathcal{Z}_{K*L}(X, A) \cong \mathcal{Z}_K(X, A) \times \mathcal{Z}_L(X, A)$.
- Takes a cellular pair (X, A) to a cellular subcomplex of $X^{\times m}$.
- Particular case: $\mathcal{Z}_K(X) := \mathcal{Z}_K(X, *)$.

Functoriality properties

- Let $f\colon (X,A)\to(Y,B)$ be a (cellular) map. Then $f^{\times n}\colon X^{\times n}\to Y^{\times n}$ restricts to a (cellular) map $\mathcal{Z}_K(f)\colon \mathcal{Z}_K(X,A)\to\mathcal{Z}_K(Y,B)$.
- Let $f\colon (X,*)\hookrightarrow(Y,*)$ be a cellular inclusion. Then, $\mathcal{Z}_K(f)_*\colon C_q(\mathcal{Z}_K(X))\hookrightarrow C_q(\mathcal{Z}_K(Y))$ admits a retraction, $\forall q\geqslant 0$.
- Let $\phi\colon K\hookrightarrow L$ be the inclusion of a full subcomplex. Then there are induced maps $\mathcal{Z}^{\phi}\colon \mathcal{Z}_L(X,A)\twoheadrightarrow\mathcal{Z}_K(X,A)$ and $\mathcal{Z}_{\phi}\colon \mathcal{Z}_K(X,A)\hookrightarrow\mathcal{Z}_L(X,A)$, such that $\mathcal{Z}_{\phi}\circ\mathcal{Z}^{\phi}=\mathrm{id}$.

Fundamental group and asphericity (Davis)

- $\pi_1(\mathcal{Z}_K(X,*))$ is the graph product of $G_v=\pi_1(X,*)$ along the graph $\Gamma=K^{(1)}$, where

$$\mathrm{Prod}_{\Gamma}(G_v)=\mathop{\ast}_{v\in V} G_v/\{[g_v,g_w]=1 \text{ if } \{v,w\}\in E,\ g_v\in G_v,\ g_w\in G_w\}.$$

- Suppose X is aspherical. Then $\mathcal{Z}_K(X)$ is aspherical iff K is a flag complex.

Generalized Davis–Januszkiewicz spaces

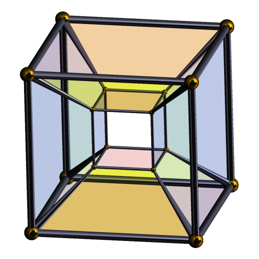
- G abelian topological group $G \quad \rightsquigarrow \quad$ GDJ space $\mathcal{Z}_K(BG)$.
- We have a bundle $G^m \to \mathcal{Z}_K(EG, G) \to \mathcal{Z}_K(BG)$.
- If G is a finitely generated (discrete) abelian group, then $\pi_1(\mathcal{Z}_K(BG))_{\text{ab}} = G^m$, and thus $\mathcal{Z}_K(EG, G)$ is the universal abelian cover of $\mathcal{Z}_K(BG)$.
- $G = S^1$: Usual Davis–Januszkiewicz space, $\mathcal{Z}_K(\mathbb{CP}^\infty)$.
  - $\pi_1 = \{1\}$.
  - $H^*(\mathcal{Z}_K(\mathbb{CP}^\infty), \mathbb{Z}) = S/I_K$, where $S = \mathbb{Z}[x_1, \dots, x_m]$, $\deg x_i = 2$.
- $G = \mathbb{Z}_2$: Real Davis–Januszkiewicz space, $\mathcal{Z}_K(\mathbb{RP}^\infty)$.
  - $\pi_1 = W_K$, the right-angled Coxeter group associated to $K^{(1)}$.
  - $H^*(\mathcal{Z}_K(\mathbb{RP}^\infty), \mathbb{Z}_2) = R/I_K$, where $R = \mathbb{Z}_2[x_1, \dots, x_m]$, $\deg x_i = 1$.
- $G = \mathbb{Z}$: Toric complex, $\mathcal{Z}_K(S^1)$.
  - $\pi_1 = A_K$, the right-angled Artin group associated to $K^{(1)}$.
  - $H^*(\mathcal{Z}_K(S^1), \mathbb{Z}) = E/J_K$, where $E = \bigwedge[e_1, \dots, e_m]$, $\deg e_i = 1$.

Standard moment-angle complexes

- Complex moment-angle complex, $\mathcal{Z}_K(D^2, S^1) \simeq \mathcal{Z}_K(ES^1, S^1)$.
  - $\pi_1 = \pi_2 = \{1\}$.
  - $H^*(\mathcal{Z}_K(D^2, S^1), \mathbb{Z}) = \operatorname{Tor}^S(S/I_K, \mathbb{Z})$.
- Real moment-angle complex, $\mathcal{Z}_K(D^1, S^0) \simeq \mathcal{Z}_K(E\mathbb{Z}_2, \mathbb{Z}_2)$.
  - $\pi_1 = W'_K$, the derived subgroup of W_K.
  - $H^*(\mathcal{Z}_K(D^1, S^0), \mathbb{Z}_2) = \operatorname{Tor}^R(R/I_K, \mathbb{Z}_2)$ — only additively!

Example

Let K be a circuit on 4 vertices.
Then $\mathcal{Z}_K(D^2, S^1) = S^3 \times S^3$,
while $\mathcal{Z}_K(D^1, S^0) = S^1 \times S^1$
(embedded in the 4-cube).



Theorem (Bahri, Bendersky, Cohen, Gitler)

Let K a simplicial complex on m vertices. There is a natural homotopy equivalence

$$\Sigma(\mathcal{Z}_K(X,A)) \simeq \Sigma\Big(\bigvee_{I\subset[m]} \widehat{\mathcal{Z}}_{K_I}(X,A)\Big),$$

where K_I is the induced subcomplex of K on the subset $I \subset [m]$.

Corollary

If X is contractible and A is a discrete subspace consisting of p points, then

$$H_k(\mathcal{Z}_K(X,A);R) \cong \bigoplus_{I\subset[m]} \bigoplus_{1}^{(p-1)^{|I|}} \widetilde{H}_{k-1}(K_I;R).$$

Finite abelian covers

- Let X be a connected, finite-type CW-complex, with $\pi = \pi_1(X, x_0)$.
- Let $p\colon Y \to X$ a (connected) regular cover, with group of deck transformations Γ. We then have a short exact sequence

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_\sharp} \pi_1(X, x_0) \xrightarrow{\nu} \Gamma \longrightarrow 1 .$$

- Conversely, every epimorphism $\nu\colon \pi \twoheadrightarrow \Gamma$ defines a regular cover $X^\nu \to X$ (unique up to equivalence), with $\pi_1(X^\nu) = \ker(\nu)$.
- If Γ is abelian, then $\nu = \chi \circ \mathrm{ab}$ factors through the abelianization, while $X^\nu = X^\chi$ is covered by the universal abelian cover of X:

$$\begin{array}{ccc} X^{\mathrm{ab}} & \longrightarrow & X^\nu \\ & \searrow & \downarrow p \\ & & X \end{array} \qquad \longleftrightarrow \qquad \begin{array}{ccc} \pi_1(X) & \xrightarrow{\mathrm{ab}} & \pi_1(X)_{\mathrm{ab}} \\ & \searrow^{\nu} & \downarrow \chi \\ & & \Gamma \end{array}$$

- Let $C_q(X^\nu;\Bbbk)$ be the group of cellular q-chains on X^ν, with coefficients in a field $\Bbbk$. We then have natural isomorphisms

$$C_q(X^\nu;\Bbbk) \cong C_q(X;\Bbbk\Gamma) \cong C_q(\widetilde{X}) \otimes_{\Bbbk\pi} \Bbbk\Gamma.$$

- Now suppose Γ is finite abelian, $\Bbbk = \bar{\Bbbk}$, and $\operatorname{char}\Bbbk = 0$. Then, all $\Bbbk$-irreps of Γ are 1-dimensional, and so

$$C_q(X^\nu;\Bbbk) \cong \bigoplus_{\rho\in\operatorname{Hom}(\Gamma,\Bbbk^\times)} C_q(X;\Bbbk_{\rho\circ\nu}),$$

  where $\Bbbk_{\rho\circ\nu}$ denotes the field $\Bbbk$, viewed as a $\Bbbk\pi$-module via the character $\rho\circ\nu\colon \pi\to\Bbbk^\times$.

- Thus, $H_q(X^\nu;\Bbbk) \cong \bigoplus_{\rho\in\operatorname{Hom}(\Gamma,\Bbbk^\times)} H_q(X;\Bbbk_{\rho\circ\nu})$.

- Now let P be an n-dimensional, simple polytope with m facets, and set $K = K_{\partial P}$.
- Let $\chi\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a characteristic matrix for P.
- Then $\ker(\chi) \cong \mathbb{Z}_2^{m-n}$ acts freely on $\mathcal{Z}_K(D^1, S^0)$, with quotient the real quasi-toric manifold $N_P(\chi)$.
- $N_P(\chi)$ comes equipped with an action of $\mathbb{Z}_2^m / \ker(\chi) \cong \mathbb{Z}_2^n$; the orbit space is P.
- Furthermore, $\mathcal{Z}_K(D^1, S^0)$ is homotopy equivalent to the maximal abelian cover of $\mathcal{Z}_K(\mathbb{RP}^\infty)$, corresponding to the sequence

$$1 \longrightarrow W'_K \longrightarrow W_K \xrightarrow{\text{ab}} \mathbb{Z}_2^m \longrightarrow 1 \,.$$

- Thus, $N_P(\chi)$ is, up to homotopy, a regular $\mathbb{Z}_2^n$-cover of $\mathcal{Z}_K(\mathbb{RP}^\infty)$, corresponding to the sequence

$$1 \longrightarrow \pi_1(N_P(\chi)) \longrightarrow W_K \xrightarrow{\nu=\chi\circ\text{ab}} \mathbb{Z}_2^n \longrightarrow 1 \,.$$

The homology of abelian covers of GDJ spaces

- Let K be a simplicial complex on m vertices.
- Identify $\pi_1(\mathcal{Z}_K(B\mathbb{Z}_p))_{\text{ab}} = \mathbb{Z}_p^m$, generated by $x_1, \dots, x_m$.
- Let $\lambda\colon \mathbb{Z}_p^m \to \Bbbk^\times$ be a character, $\operatorname{supp}(\lambda) = \{i \in [m] \mid \lambda(x_i) \neq 1\}$.
- Let K_λ be the induced subcomplex on vertex set $\operatorname{supp}(\lambda)$.

Proposition

$$H_q(\mathcal{Z}_K(B\mathbb{Z}_p); \Bbbk_\lambda) \cong \widetilde{H}_{q-1}(K_\lambda; \Bbbk).$$

Idea: The inclusion $\iota\colon (S^1, *) \hookrightarrow (B\mathbb{Z}_p, *)$ induces a cellular inclusion $\mathcal{Z}_K(\iota)\colon T_K = \mathcal{Z}_K(S^1) \hookrightarrow \mathcal{Z}_K(B\mathbb{Z}_p)$. Moreover, $\phi\colon K_\lambda \hookrightarrow K$ induces a cellular inclusion $\mathcal{Z}_\phi\colon T_{K_\lambda} \hookrightarrow T_K$. Let $\bar{\lambda}\colon \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}_p^m \xrightarrow{\lambda} \Bbbk^\times$. We then get (chain) retractions

$$\begin{array}{ccccc} & & C_q(T_K; \Bbbk_{\bar{\lambda}}) & & \\ & \nearrow & \downarrow & & \\ C_q(\mathcal{Z}_K(B\mathbb{Z}_p); \Bbbk_\lambda) & \longrightarrow & C_q(T_{K_\lambda}; \Bbbk_{\bar{\lambda}}) & \xrightarrow{\cong} & \widetilde{C}_{q-1}(K_\lambda; \Bbbk) \end{array}$$

This shows that $\dim_{\Bbbk} H_q(\mathcal{Z}_K(B\mathbb{Z}_p);\Bbbk_\lambda) \geqslant \dim_{\Bbbk} \widetilde{H}_{q-1}(K_\lambda;\Bbbk)$.
For the reverse inequality, we use [BBCG], which, in this case, says

$$H_q(\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p);\Bbbk) \cong \bigoplus_{I\subset[m]} \bigoplus_{1}^{(p-1)^{|I|}} \widetilde{H}_{q-1}(K_I;\Bbbk),$$

and the fact that $\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p) \simeq (\mathcal{Z}_K(B\mathbb{Z}_p))^{\text{ab}}$, which gives

$$H_q(\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p);\Bbbk) \cong \bigoplus_{\rho\in\operatorname{Hom}(\mathbb{Z}_p^m,\Bbbk^\times)} H_q(\mathcal{Z}_K(B\mathbb{Z}_p);\Bbbk_\rho).$$

Theorem

Let G be a prime-order cyclic group, and let $\mathcal{Z}_K(BG)^\chi$ be the abelian cover defined by an epimorphism $\chi\colon G^m \twoheadrightarrow \Gamma$. Then

$$H_q(\mathcal{Z}_K(BG)^\chi;\Bbbk) \cong \bigoplus_{\rho\in\operatorname{Hom}(\Gamma;\Bbbk^\times)} \widetilde{H}_{q-1}(K_{\rho\circ\chi};\Bbbk),$$

where $K_{\rho\circ\chi}$ is the induced subcomplex of K on vertex set $\operatorname{supp}(\rho\circ\chi)$.

The homology of real quasi-toric manifolds

- Let again P be a simple polytope, and set $K = K_{\partial P}$.
- Let $\chi \colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a characteristic matrix for P.
- Denote by $\chi_i \in \mathbb{Z}_2^m$ the i-th row of χ.
- For each subset $S \subseteq [n]$, write $\chi_S = \sum_{i \in S} \chi_i \in \mathbb{Z}_2^m$.
- S also determines a character $\rho_S \colon \mathbb{Z}_2^n \to \Bbbk^{\times}$, taking the i-th generator to -1 if $i \in S$, and to 1 if $i \notin S$.
- Every $\rho \in \operatorname{Hom}(\mathbb{Z}_2^n, \mathbb{C}^{\times})$ arises as $\rho = \rho_S$, where $S = \operatorname{supp}(\rho)$.
- $\operatorname{supp}(\rho_S \circ \chi)$ consists of those $j \in [m]$ for which the j-th entry of χ_S is non-zero.
- Let $K_{\chi,S}$ be the induced subcomplex on this vertex set.

Corollary

The Betti numbers of the real, quasi-toric manifold $N_P(\chi)$ are given by

$$b_q(N_P(\chi)) = \sum_{S \subseteq [n]} \tilde{b}_{q-1}(K_{\chi,S}).$$

Example

- Again, let P be the square, and $K = K_{\partial P}$ the 4-cycle.
- Let $T^2 = N_P(\chi)$, where $\chi = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.
- Compute:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
χ_S	$(0\ 0\ 0\ 0)$	$(1\ 0\ 1\ 0)$	$(0\ 1\ 0\ 1)$	$(1\ 1\ 1\ 1)$
$\mathrm{supp}(\chi_S)$	$\emptyset$	$\{1,3\}$	$\{2,4\}$	$\{1,2,3,4\}$
$K_{\chi,S}$	$\emptyset$	$\{\{1\},\{3\}\}$	$\{\{2\},\{4\}\}$	K

- Thus:

$$b_0(T^2) = \tilde{b}_{-1}(\emptyset) = 1,$$
$$b_1(T^2) = \tilde{b}_0(K_{\chi,\{1\}}) + \tilde{b}_0(K_{\chi,\{2\}}) = 1 + 1 = 2,$$
$$b_2(T^2) = \tilde{b}_1(K) = 1.$$

Cup products in abelian covers of GDJ-spaces

As before, let $X^\nu \to X$ be a regular, finite abelian cover, corresponding to an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \Gamma$, and let $\Bbbk = \mathbb{C}$. The cellular cochains on X^ν decompose as

$$C^q(X^\nu; \Bbbk) \cong \bigoplus_{\rho \in \mathrm{Hom}(\Gamma, \Bbbk^\times)} C^q(X; \Bbbk_{\rho\circ\nu}),$$

The cup product map, $C^p(X^\nu, \Bbbk) \otimes_{\Bbbk} C^q(X^\nu, \Bbbk) \xrightarrow{\smile} C^{p+q}(X^\nu, \Bbbk)$, restricts to those pieces, as follows:

$$\begin{array}{ccc}
C^p(X; \Bbbk_{\rho\circ\nu}) \otimes_{\Bbbk} C^q(X; \Bbbk_{\rho'\circ\nu}) & \xrightarrow{\quad\smile\quad} & C^{p+q}(X; \Bbbk_{(\rho\cdot\rho')\circ\nu}) \\
\Big\downarrow \cong & & \Big\uparrow \Delta^* \\
C^{p+q}(X \times X; \Bbbk_{\rho\circ\nu} \otimes_{\Bbbk} \Bbbk_{\rho'\circ\nu}) & \xrightarrow{\ \mu^*\ } & C^{p+q}(X \times X; \Bbbk_{(\rho\otimes\rho')\circ\nu})
\end{array}$$

where μ^* is induced by the multiplication map on coefficients, and Δ^* is induced by a cellular approximation to the diagonal $\Delta\colon X \to X \times X$.

Proposition

Let $\mathcal{Z}_K(B\mathbb{Z}_p)^\nu$ be a regular abelian cover, with characteristic homomorphism $\chi\colon \mathbb{Z}_p^m \to \Gamma$. The cup product in

$$H^*(\mathcal{Z}_K(BG)^\nu;\Bbbk) \cong \bigoplus_{q=0}^{\infty} \left(\bigoplus_{\rho \in \operatorname{Hom}(\Gamma;\Bbbk^\times)} \widetilde{H}^{q-1}(K_{\rho\circ\chi};\Bbbk) \right)$$

is induced by the following maps on simplicial cochains:

$$\begin{aligned}
\widetilde{C}^{p-1}(K_{\rho\circ\chi};\Bbbk^\times) \otimes \widetilde{C}^{q-1}(K_{\rho'\circ\chi};\Bbbk^\times) &\to \widetilde{C}^{p+q-1}(K_{(\rho\otimes\rho')\circ\chi};\Bbbk^\times)\\
\hat{\sigma}\otimes\hat{\tau} &\mapsto \begin{cases} \pm\widehat{\sigma\sqcup\tau} & \text{if } \sigma\cap\tau=\emptyset,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

where $\sigma\sqcup\tau$ is the simplex with vertex set the union of the vertex sets of σ and τ, and $\hat{\sigma}$ is the Kronecker dual of σ.

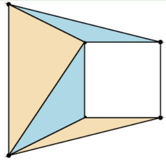
Formality properties

- A finite-type CW-complex X is *formal* if its Sullivan minimal model is quasi-isomorphic to $(H^*(X,\mathbb{Q}),0)$—roughly speaking, $H^*(X,\mathbb{Q})$ determines the rational homotopy type of X.
- (Notbohm–Ray) If X is formal, then $\mathcal{Z}_K(X)$ is formal.
- In particular, toric complexes $T_K=\mathcal{Z}_K(S^1)$ and generalized Davis–Januszkiewicz spaces $\mathcal{Z}_K(BG)$ are always formal.
- (Félix, Tanré) More generally, if both X and A are formal, and the inclusion $i\colon A\hookrightarrow X$ induces a surjection $i^*\colon H^*(X,\mathbb{Q})\to H^*(A,\mathbb{Q})$, then $\mathcal{Z}_K(X,A)$ is formal.
- (Panov–Ray) Complex quasi-toric manifolds $M_P(\chi)$ are always formal.

- (Baskakov, Denham–A.S.) Moment angle complexes $\mathcal{Z}_K(D^2, S^1)$ are not always formal: they can have non-zero Massey products.

Example (D-S)

A simplicial complex K
for which $\mathcal{Z}_K(D^2, S^1)$ carries
a non-trivial triple Massey product.



- It follows that real moment-angle complexes $\mathcal{Z}_K(D^1, S^0)$ are not always formal.
- Question: are real quasi-toric manifolds $N_P(\chi)$ formal?