Milnor fibrations of hyperplane arrangements

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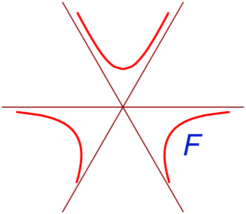
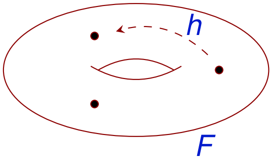
THE MILNOR FIBRATION

- Let $f \in \mathbb{C}[z_0, \dots, z_d]$ be a homogeneous polynomial of degree n.
- Let $V(f) = \{z \in \mathbb{C}^{d+1} \mid f(z) = 0\}$ and $M = \mathbb{C}^{d+1} \setminus V(f)$.
- The map $f \colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a map $f \colon M \to \mathbb{C}^*$.
- This is the projection of a smooth, locally trivial bundle, known as the (global) *Milnor fibration* of f.
- The typical fiber, $F = f^{-1}(1)$, is homotopic to a finite CW-complex of dim d. If f is not a proper power, then F is connected.
- The monodromy of the fibration: $h \colon F \to F$, $z \mapsto e^{2\pi \mathrm{i}/n} z$.
- The algebraic monodromy: $h_q \colon H_q(F, \mathbb{C}) \to H_q(F, \mathbb{C})$.

- If f has an isolated critical point at 0, then $F \simeq \bigvee^{\mu} S^d$, where $\mu = (n-1)^{d+1}$.
- For instance, let $f = z_0^3 - z_1^3$. Then F is a thrice-punctured torus (with h rotation by 120°), and $F \simeq \bigvee^4 S^1$:





- More generally, if $f = z_0^n - z_1^n$, then F is a Riemann surface of genus $\binom{n-1}{2}$ with n punctures, and so $F \simeq \bigvee^{(n-1)^2} S^1$.
- If the singularity at 0 is non-isolated, though, the Betti numbers $b_q(F)$ and the algebraic monodromies h_q are hard to compute.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite cell complex, and let $\pi = \pi_1(X, x_0)$.

- Let $\Bbbk$ be an algebraically closed field, and let $\mathrm{Hom}(\pi, \Bbbk^*) = H^1(X, \Bbbk^*)$ be the character group of π.

- The (degree 1) *characteristic varieties* of X are the jump loci for homology with coefficients in rank-1 local systems on X:

$$\mathcal{V}_s(X, \Bbbk) = \{\rho \in \mathrm{Hom}(\pi, \Bbbk^*) \mid \dim_{\Bbbk} H_1(X, \Bbbk_\rho) \geqslant s\}.$$

EXAMPLE (CIRCLE)

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\Bbbk\mathbb{Z} = \Bbbk[t^{\pm 1}]$.
Then:

$$C_*(\widetilde{S^1}, \Bbbk) : \; 0 \longrightarrow \Bbbk[t^{\pm 1}] \xrightarrow{t-1} \Bbbk[t^{\pm 1}] \longrightarrow 0 \, .$$

For $\rho \in \operatorname{Hom}(\mathbb{Z}, \Bbbk^*) = \Bbbk^*$, we get

$$C_*(\widetilde{S^1}, \Bbbk) \otimes_{\Bbbk\mathbb{Z}} \Bbbk_\rho : \; 0 \longrightarrow \Bbbk \xrightarrow{\rho-1} \Bbbk \longrightarrow 0 \, ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \Bbbk) = H_1(S^1, \Bbbk) = \Bbbk$.
Hence: $\mathcal{V}_1(S^1, \Bbbk) = \{1\}$ and $\mathcal{V}_s(S^1, \Bbbk) = \emptyset$, otherwise.

EXAMPLE (PUNCTURED COMPLEX LINE)

Identify $\pi_1(\mathbb{C} \backslash \{n \text{ points}\}) = F_n$, and $\operatorname{Hom}(F_n, \Bbbk^*) = (\Bbbk^*)^n$. Then:

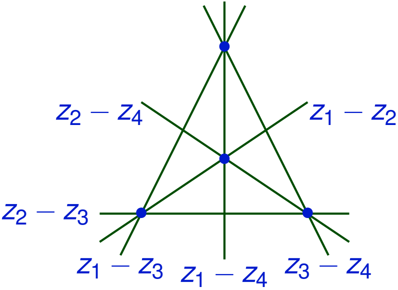
$$\mathcal{V}_s(\mathbb{C} \backslash \{n \text{ points}\}, \Bbbk) = \begin{cases} (\Bbbk^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \emptyset & \text{if } s > n. \end{cases}$$

- Let $\pi\colon \mathbb{C}^{d+1}\backslash\{0\} \to \mathbb{CP}^d$ be the projection map, with fiber $\mathbb{C}^*$.
- This map restricts to $\pi\colon M \to U$, where $U = M/\mathbb{C}^* = \mathbb{CP}^d\backslash V(f)$.
- This map further restricts to a regular, $\mathbb{Z}_n$-cover $F \to U$.
- Assume f is square-free, and write $f = f_1 \cdots f_r$, with factors irreducible and distinct.
- Then the cover $F \to U$ is classified by the homomorphism $\delta\colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_n$ that sends each meridian about $V(f_i)$ to $\deg(f_i)$.
- Fix a field $\Bbbk$, and let $\hat{\delta}\colon \operatorname{Hom}(\mathbb{Z}_n, \Bbbk^*) \to \operatorname{Hom}(\pi_1(U), \Bbbk^*)$ be the induced homomorphism between character groups.
- If $\operatorname{char}(\Bbbk) \nmid n$, then

$$\dim_{\Bbbk} H_1(F, \Bbbk) = \sum_{s\geqslant 1} \left| \mathcal{V}_s(U, \Bbbk) \cap \operatorname{im}(\hat{\delta}) \right| .$$

HYPERPLANE ARRANGEMENTS

- $\mathcal{A}$: A (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$.
- Intersection lattice: $L(\mathcal{A})$.
- Complement: $M(\mathcal{A}) = \mathbb{C}^{\ell} \setminus \bigcup_{H\in\mathcal{A}} H$.
- The Boolean arrangement $\mathcal{B}_n$
  - $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{B}_n)$: lattice of subsets of $\{0, 1\}^n$.
  - $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n$.
- The braid arrangement $\mathcal{A}_n$ (or, reflection arr. of type A_{n-1})
  - $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
  - $L(\mathcal{A}_n)$: lattice of partitions of $[n] = \{1, \dots, n\}$.
  - $M(\mathcal{A}_n)$: configuration space of n ordered points in $\mathbb{C}$ (a classifying space for the pure braid group on n strings).



- M has the homotopy type of a connected, finite CW-complex of dimension $d+1$. In fact, M admits a minimal cell structure.
- In particular, $H_*(M,\mathbb{Z})$ is torsion-free. The Betti numbers $b_q(M) := \operatorname{rank} H_q(M,\mathbb{Z})$ are given by the Möbius function of $L(\mathcal{A})$.
- The Orlik–Solomon algebra $A = H^*(M,\mathbb{Z})$ is determined by $L(\mathcal{A})$. but $\pi_1(M)$ is not.

MILNOR FIBRATIONS OF ARRANGEMENTS

- For each $H \in \mathcal{A}$, let $f_H \colon \mathbb{C}^{d+1} \to \mathbb{C}$ be a linear form with kernel H.
- Let $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} f_H$, a homogeneous polynomial of degree n.
- This polynomial defines the Milnor fibration of $\mathcal{A}$, with fiber $F = F(\mathcal{A})$.

EXAMPLE

Let $\mathcal{B}_n$ be the Boolean arrangement, with $Q = z_1 \cdots z_n$. Then $M(\mathcal{B}_n) = (\mathbb{C}^*)^n$ and $F(\mathcal{B}_n) = \ker(\mathbb{Q}) \cong (\mathbb{C}^*)^{n-1}$.

- Let $\mathcal{A}$ be an arrangement of planes in $\mathbb{C}^3$. Its projectivization, $\bar{\mathcal{A}}$, is an arrangement of lines in $\mathbb{CP}^2$.
- A flat $X \in L_2(\mathcal{A})$ has multiplicity q if $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \supset H\}$ has size q, i.e., there are exactly q lines from $\bar{\mathcal{A}}$ passing through $\bar{X}$.

Question: Are the Betti numbers of $F(\mathcal{A})$ and the characteristic polynomial of the algebraic monodromy determined by $L(\mathcal{A})$? Let $\Delta_{\mathcal{A}}(t) := \det(h_1 - t \cdot \mathrm{id})$. Then $b_1(F(\mathcal{A})) = \deg \Delta_{\mathcal{A}}$.

THEOREM

Suppose all flats $X \in L_2(\mathcal{A})$ have multiplicity 2 or 3. Then $\Delta_{\mathcal{A}}(t)$, and thus $b_1(F(\mathcal{A}))$, are combinatorially determined.

- We relate the cohomology jump loci of $M(\mathcal{A})$ in characteristic p with those in characteristic 0.
- A key combinatorial ingredient is the notion of multinet.

RESONANCE VARIETIES AND THE β_p-INVARIANTS

- Let $A = H^*(M(\mathcal{A}), \Bbbk)$ — an algebra that depends only on $L(\mathcal{A})$ (and the field $\Bbbk$).
- For each $a \in A^1$, we have $a^2 = 0$. Thus, we get a cochain complex, $(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$
- The (degree 1) *resonance varieties* of $\mathcal{A}$ are the cohomology jump loci of this “Aomoto complex":

$$\mathcal{R}_s(\mathcal{A}, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^1(A, \cdot a) \geqslant s\},$$

- In particular, $a \in A^1$ belongs to $\mathcal{R}_1(\mathcal{A}, \Bbbk)$ iff there is $b \in A^1$ not proportional to a, such that $a \cup b = 0$ in A^2.

- Now assume $\Bbbk$ has characteristic $p > 0$.

- Let $\sigma = \sum_{H \in \mathcal{A}} e_H \in A^1$ be the "diagonal" vector, and define

$$\beta_p(\mathcal{A}) = \dim_{\Bbbk} H^1(A, \cdot\sigma).$$

That is, $\beta_p(\mathcal{A}) = \max\{s \mid \sigma \in \mathcal{R}^1_s(A, \Bbbk)\}$.

- Clearly, $\beta_p(\mathcal{A})$ depends only on $L(\mathcal{A})$ and p. Moreover, $0 \leqslant \beta_p(\mathcal{A}) \leqslant |\mathcal{A}| - 2$.

THEOREM

If $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$, then $\beta_3(\mathcal{A}) \leqslant 2$.

- Let $\Bbbk = \mathbb{F}_{p^s}$ be a finite field different from $\mathbb{F}_2$.
- Let $\mathcal{M}_{\Bbbk}(m)$ be the simple matroid on $\Bbbk^m$ whose size 3 dependent subsets are all 3-tuples $\{v, v', v''\}$ for which $v + v' + v'' = 0$.
- $\mathcal{M}_{\Bbbk}(m)$ has rank 2 for $m = 1$, and rank 3 for $m > 1$. Moreover all 2-flats have multiplicity p^s.

THEOREM

1. *$\beta_p(\mathcal{M}_{\Bbbk}(m)) \geqslant m$, while $\beta_3(\mathcal{M}_{\mathbb{F}_3}(m)) = m$, for all $m \geqslant 1$.*
2. *If $\Bbbk \neq \mathbb{F}_3$, the matroids $\mathcal{M}_{\Bbbk}(m)$ are non-realizable over $\mathbb{C}$, for all $m \geqslant 2$.*
3. *If $\Bbbk = \mathbb{F}_3$, the matroids $\mathcal{M}_{\Bbbk}(m)$ are non-realizable over $\mathbb{C}$, for all $m \geqslant 3$.*

- (2) follows from the Hirzebruch–Miyaoka–Yau inequality.
- (3) is more subtle: it uses a result of Yuzvinsky on 3-nets.

THE HOMOLOGY OF THE MILNOR FIBER

- The monodromy $h\colon F(\mathcal{A}) \to F(\mathcal{A})$ has order $n = |\mathcal{A}|$. Thus,

$$\Delta_{\mathcal{A}}(t) = \prod_{d|n} \Phi_d(t)^{e_d(\mathcal{A})},$$

  where $\Phi_1 = t-1$, $\Phi_2 = t+1$, $\Phi_3 = t^2+t+1$, $\Phi_4 = t^2+1$, … are the cyclotomic polynomials, and $e_d(\mathcal{A}) \in \mathbb{Z}_{\geqslant 0}$.

- Easy to see: $e_1(\mathcal{A}) = n-1$. Hence, $H_1(F(\mathcal{A}), \mathbb{C})$, when viewed as a module over $\mathbb{C}[\mathbb{Z}_n]$, decomposes as

$$(\mathbb{C}[t]/(t-1))^{n-1} \oplus \bigoplus_{1<d|n} (\mathbb{C}[t]/\Phi_d(t))^{e_d(\mathcal{A})}.$$

- In particular, $b_1(F(\mathcal{A})) = n-1+\sum_{1<d|n}\varphi(d)e_d(\mathcal{A})$.

- Thus, in degree 1, question (Q1) is equivalent to: are the integers $e_d(\mathcal{A})$ determined by $L_{\leqslant 2}(\mathcal{A})$?
- Not all divisors of n appear in the above formulas: If d does *not* divide $|\mathcal{A}_X|$, for some $X \in L_2(\mathcal{A})$, then $e_d(\mathcal{A}) = 0$ (Libgober).
- In particular, if $L_2(\mathcal{A})$ has only flats of multiplicity 2 and 3, then $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}(t^2+t+1)^{e_3}$.
- If multiplicity 4 appears, then also get factor of $(t+1)^{e_2} \cdot (t^2+1)^{e_4}$.

THEOREM (COHEN–ORLIK 2000, PAPADIMA–S. 2010)

$e_{p^s}(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$, for all $s \geqslant 1$.

THEOREM

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $e_3(\mathcal{A}) = \beta_3(\mathcal{A})$, and thus $e_3(\mathcal{A})$ is combinatorially determined.

A similar result holds for $e_2(\mathcal{A})$ and $e_4(\mathcal{A})$, under some additional hypothesis.

COROLLARY

If $\bar{\mathcal{A}}$ is an arrangement of n lines in $\mathbb{P}^2$ with only double and triple points, then $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}(t^2+t+1)^{\beta_3(\mathcal{A})}$ is combinatorially determined.

COROLLARY (LIBGOBER 2012)

If $\bar{\mathcal{A}}$ is an arrangement of n lines in $\mathbb{P}^2$ with only double and triple points, then the question whether $\Delta_{\mathcal{A}}(t) = (t-1)^{n-1}$ or not is combinatorially determined.

CONJECTURE

Let $\mathcal{A}$ be an arrangement of rank at least 3. Then

$$e_{p^s}(\mathcal{A}) = 0$$

for all primes p and integers $s \geqslant 1$, with two possible exceptions:

$$e_2(\mathcal{A}) = e_4(\mathcal{A}) = \beta_2(\mathcal{A}) \text{ and } e_3(\mathcal{A}) = \beta_3(\mathcal{A}).$$

If $e_d(\mathcal{A}) = 0$ for all divisors d of $|\mathcal{A}|$ which are not prime powers, this conjecture would give:

$$\Delta_{\mathcal{A}}(t) = (t-1)^{|\mathcal{A}|-1}((t+1)(t^2+1))^{\beta_2(\mathcal{A})}(t^2+t+1)^{\beta_3(\mathcal{A})}.$$

The conjecture has been verified for several classes of arrangements, including complex reflection arrangements and certain types of real arrangements.

MULTINETS

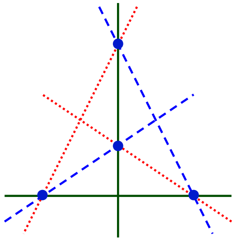
DEFINITION (FALK AND YUZVINSKY)

A *multinet* on $\mathcal{A}$ is a partition of the set $\mathcal{A}$ into $k \geqslant 3$ subsets $\mathcal{A}_1, \dots, \mathcal{A}_k$, together with an assignment of multiplicities, $m \colon \mathcal{A} \to \mathbb{N}$, and a subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus, such that:

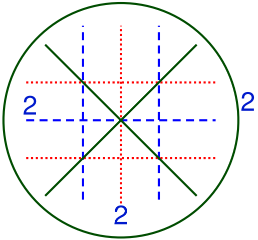
1. There is an integer d such that $\sum_{H \in \mathcal{A}_\alpha} m_H = d$, for all $\alpha \in [k]$.
2. If H and H' are in different classes, then $H \cap H' \in \mathcal{X}$.
3. For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_\alpha : H \supset X} m_H$ is independent of α.
4. Each set $\left(\bigcup_{H \in \mathcal{A}_\alpha} H\right) \backslash \mathcal{X}$ is connected.

- A similar definition can be made for any (rank 3) matroid.
- A multinet as above is also called a (k, d)-multinet, or a k-multinet.
- The multinet is *reduced* if $m_H = 1$, for all $H \in \mathcal{A}$.

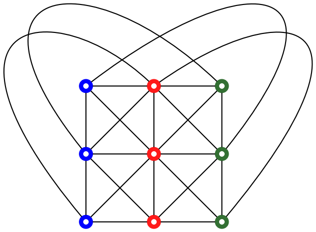
- A *net* is a reduced multinet with $n_X = 1$, for all $X \in \mathcal{X}$.
- In this case, $|\mathcal{A}_\alpha| = |\mathcal{A}| / k = d$, for all α.
- Moreover, $\bar{\mathcal{X}}$ has size d^2, and is encoded by a $(k-2)$-tuple of orthogonal Latin squares.

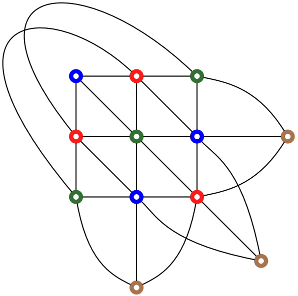


A $(3,2)$-net on the A_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)



A $(3,4)$-multinet on the B_3 arrangement
$\bar{\mathcal{X}}$ consists of 4 triple points ($n_X = 1$)
and 3 triple points ($n_X = 2$)





A $(3,3)$-net on the Ceva matroid. A $(4,3)$-net on the Hessian matroid.

- If $\mathcal{A}$ has no flats of multiplicity kr, for some $r > 1$, then every reduced k-multinet is a k-net.
- (Kawahara): given any Latin square, there is a matroid $\mathcal{M}$ with a 3-net $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ realizing it, such that each $\mathcal{M}_\alpha$ is uniform.
- (Yuzvinsky and Pereira–Yuz): If $\mathcal{A}$ supports a k-multinet with $|\mathcal{X}| > 1$, then $k = 3$ or 4; if the multinet is not reduced, then $k = 3$.
- (Wakefield & al): The only $(4, 3)$-net in $\mathbb{CP}^2$ is the Hessian; there are no $(4, 4)$, $(4, 5)$, or $(4, 6)$ nets in $\mathbb{CP}^2$.
- Conjecture (Yuz): The only 4-multinet is the Hessian $(4, 3)$-net.
- (Torielli–Yoshinaga): There are no 4-nets on real arrangements.

LEMMA

If $\mathcal{A}$ supports a 3-net with parts $\mathcal{A}_\alpha$, then:

1. $1 \leqslant \beta_3(\mathcal{A}) \leqslant \beta_3(\mathcal{A}_\alpha) + 1$, *for all* α.
2. *If* $\beta_3(\mathcal{A}_\alpha) = 0$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$.
3. *If* $\beta_3(\mathcal{A}_\alpha) = 1$, *for some* α, *then* $\beta_3(\mathcal{A}) = 1$ *or* 2.

All possibilities do occur:

- Braid arrangement: has a $(3, 2)$-net from the Latin square of $\mathbb{Z}_2$. $\beta_3(\mathcal{A}_\alpha) = 0$ $(\forall\alpha)$ and $\beta_3(\mathcal{A}) = 1$.
- Pappus arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_1) = \beta_3(\mathcal{A}_2) = 0$, $\beta_3(\mathcal{A}_3) = 1$ and $\beta_3(\mathcal{A}) = 1$.
- Ceva arrangement: has a $(3, 3)$-net from the Latin square of $\mathbb{Z}_3$. $\beta_3(\mathcal{A}_\alpha) = 1$ $(\forall\alpha)$ and $\beta_3(\mathcal{A}) = 2$.

Complex cohomology jump loci

Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Work of Arapura, Falk, Cohen–S., Libgober–Yuz, Falk–Yuz completely describes the varieties $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$:

- $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ is a union of linear subspaces in $H^1(M(\mathcal{A}), \mathbb{C}) = \mathbb{C}^{|\mathcal{A}|}$.
- Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$ is the union of those linear subspaces that have dimension at least $s + 1$.

- Each flat $X \in L_2(\mathcal{A})$ of multiplicity $k \geqslant 3$ gives rise to a *local* component of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$, of dimension $k - 1$.
- More generally, every k-multinet on a sub-arrangement $\mathcal{B} \subseteq \mathcal{A}$ gives rise to a component of dimension $k - 1$, and all components of $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ arise in this way.
- Note: the varieties $\mathcal{R}_1(\mathcal{A}, \Bbbk)$ with $\operatorname{char}(\Bbbk) > 0$ can be more complicated: components may be non-linear, and they may intersect non-transversely.

THEOREM

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$, with $r > 1$. Then $\mathcal{R}_1(\mathcal{A}, \mathbb{C})$ has at least $(3^{\beta_3(\mathcal{A})} - 1)/2$ essential components, all corresponding to 3-nets.

Work of Arapura, Libgober, Cohen–S., S., Libgober–Yuz, Falk–Yuz, Dimca, Dimca–Papadima–S., Artal–Cogolludo–Matei, Budur–Wang ... provides a fairly explicit description of the varieties $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$:

- Each variety $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$ is a finite union of torsion-translates of algebraic subtori of $(\mathbb{C}^*)^n$.
- If a linear subspace $L \subset \mathbb{C}^n$ is a component of $\mathcal{R}_s(\mathcal{A}, \mathbb{C})$, then the algebraic torus $T = \exp(L)$ is a component of $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$.
- Moreover, $T = f^*(H^1(S, \mathbb{C}^*))$, where $f\colon M(\mathcal{A}) \to S$ is an orbifold fibration, with base $S = \mathbb{CP}^1 \setminus \{k \text{ points}\}$, for some $k \geqslant 3$.
- All components of $\mathcal{V}_s(\mathcal{A}, \mathbb{C})$ passing through the origin $1 \in (\mathbb{C}^*)^n$ arise in this way (and thus, are combinatorially determined).

THEOREM

If $\mathcal{A}$ admits a reduced k-multinet, then $e_k(\mathcal{A}) \geqslant k-2$.

MAIN THEOREM

THEOREM

Suppose $L_2(\mathcal{A})$ has no flats of multiplicity $3r$ with $r > 1$. Then TFAE:

1. $L_{\leqslant 2}(\mathcal{A})$ *admits a reduced* 3*-multinet.*
2. $L_{\leqslant 2}(\mathcal{A})$ *admits a* 3*-net.*
3. $\beta_3(\mathcal{A}) \neq 0$.
4. $e_3(\mathcal{A}) \neq 0$.

Moreover, $\beta_3(\mathcal{A}) \leqslant 2$ *and* $\beta_3(\mathcal{A}) = e_3(\mathcal{A})$.

- (2) $\Rightarrow$ (1): obvious.
- (1) $\Rightarrow$ (4): by above theorem.
- (4) $\Rightarrow$ (3): by modular bound $e_p(\mathcal{A}) \leqslant \beta_p(\mathcal{A})$.
- (3) $\Rightarrow$ (2): relate resonance and nets.
- $\beta_3(\mathcal{A}) \leqslant 2$: a previous theorem.
- Last assertion: put things together.