Abelian duality and propagation of resonance

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COHOMOLOGY JUMP LOCI

- Let $\Bbbk$ be an algebraically closed field.
- Let S be a commutative, finitely-generated $\Bbbk$-algebra.
- Let $\operatorname{Spec}(S) = \operatorname{Hom}_{\Bbbk\text{-alg}}(S, \Bbbk)$ be the maximal spectrum of S.
- Let

$$C: \ 0 \longrightarrow C^0 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{d_i} C^{i+1} \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0$$

  be a (bounded) cochain complex over S.
- The *cohomology jump loci* of C are defined as

$$\mathcal{V}^i(C) := \{\mathfrak{m} \in \operatorname{Spec}(S) \mid H^i(C \otimes_S S/\mathfrak{m}) \neq 0\}.$$

PROPAGATION

- The sets $V^i(C)$ depend only on the chain-homotopy equivalence class of C.
- Assume C is a cochain complex of free, finitely-generated S-modules. Then $\mathcal{V}^i(C)$ are Zariski closed subsets of $\mathrm{Spec}(S)$.
- We say the jump loci of C *propagate* if

$$\mathcal{V}^{i-1}(P) \subseteq \mathcal{V}^i(P) \qquad \text{for } 0 < i \leqslant n.$$

THE BGG CORRESPONDENCE

- Let V be a finite-dimensional $\Bbbk$-vector space.
- Fix basis $e_1, \dots, e_n$ for V, and dual basis $x_1, \dots, x_n$ for $V^\vee$.
- Let $E = \bigwedge V$ and $S = \operatorname{Sym} V^\vee$.
- Let P be a finitely-generated, graded E-module.
  - E.g., a graded, graded-commutative $\Bbbk$-algebra A ($\operatorname{char} \Bbbk \neq 2$).
- BGG yields a cochain complex of free, finitely-generated S-modules,

$$\mathbf{L}(P)\colon \quad \cdots \longrightarrow P^i \otimes_{\Bbbk} S \xrightarrow{d_i} P^{i+1} \otimes_{\Bbbk} S \longrightarrow \cdots,$$

with differentials $d_i(p \otimes s) = \sum_{j=1}^{n} e_j p \otimes x_j s$.

RESONANCE VARIETIES

- Evaluating $\mathbf{L}(P)$ at $a \in V$ gives the (Aomoto) cochain complex

$$(P, a) := \mathbf{L}(P) \otimes_S S/\mathfrak{m}_a: \quad \cdots \longrightarrow P^i \xrightarrow{\cdot a} P^{i+1} \longrightarrow \cdots$$

- The *resonance varieties* of P are the cohomology jump loci of $\mathbf{L}(P)$:

$$\mathcal{R}^i(P) := \mathcal{V}^i(\mathbf{L}(P)) = \{a \in V \mid H^i(P, a) \neq 0\}.$$

  They are closed cones inside the affine space $V = \mathrm{Spec}(S)$.

PROPAGATION OF RESONANCE

THEOREM (EISENBUD–POPESCU–YUZVINSKY 2003)

Let A be the Orlik–Solomon algebra of an arrangement. Then the resonance varieties of A propagate.

Using similar techniques, we obtain the following generalization.

THEOREM (DSY)

Suppose the $\Bbbk$-dual module, $\widehat{P}$, has a linear free resolution over E. Then the resonance varieties of P propagate.

JUMP LOCI OF SPACES

- Let X be a connected, finite CW-complex.
- Fundamental group $\pi = \pi_1(X, x_0)$: a finitely generated, discrete group, with $\pi_{ab} \cong H_1(X, \mathbb{Z})$.
- Let $S = \Bbbk[\pi_{ab}]$ and identify $\operatorname{Spec}(S)$ with the character group $\operatorname{Hom}(\pi, \Bbbk^*) = H^1(X, \Bbbk^*)$.
- The *characteristic varieties* of X are the cohomology jump loci of the free S-cochain complex $C = C^*(X^{ab}, \Bbbk)$:

$$\mathcal{V}^i(X, \Bbbk) = \{\rho \in H^1(X, \Bbbk^*) \mid H^i(X, \Bbbk_\rho) \neq 0\}.$$

- The *resonance varieties* of X are the jump loci associated to the cohomology algebra $A = H^*(X, \Bbbk)$:

$$\mathcal{R}^i(X, \Bbbk) = \{a \in H^1(X, \Bbbk) \mid H^i(A, a) \neq 0\}.$$

DUALITY SPACES

In order to study propagation of jump loci in a topological setting, we start by recalling a notion due to Bieri and Eckmann (1978).

- X is a *duality space* of dimension n if $H^i(X, \mathbb{Z}\pi) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi) \neq 0$ and torsion-free.
- Let $D = H^n(X, \mathbb{Z}\pi)$ be the dualizing $\mathbb{Z}\pi$-module. Given any $\mathbb{Z}\pi$-module A, we have $H^i(X, A) \cong H_{n-i}(X, D \otimes A)$.
- If $X = K(\pi, 1)$, then π is a duality group. If, furthermore, $D = \mathbb{Z}$, with trivial $\mathbb{Z}\pi$-action, then π is a Poincaré duality group.

ABELIAN DUALITY SPACES

We introduce an analogous notion, by replacing $\pi \rightsquigarrow \pi_{\mathrm{ab}}$.

- X is an *abelian duality space* of dimension n if $H^i(X, \mathbb{Z}\pi_{\mathrm{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}\pi_{\mathrm{ab}}) \neq 0$ and torsion-free.
- Let $B = H^n(X, \mathbb{Z}\pi_{\mathrm{ab}})$ be the dualizing $\mathbb{Z}\pi_{\mathrm{ab}}$-module. Given any $\mathbb{Z}\pi_{\mathrm{ab}}$-module A, we have $H^i(X, A) \cong H_{n-i}(X, B \otimes A)$.
- There are duality spaces which are not abelian duality spaces (e.g., Riemann surfaces of genus $g > 1$), and the other way around, too.

PROPAGATION OF JUMP LOCI

THEOREM

Let X be an abelian duality space of dimension n. If $\rho\colon \pi_1(X) \to \mathbb{k}^*$ satisfies $H^i(X, \mathbb{k}_\rho) \neq 0$, then $H^j(X, \mathbb{k}_\rho) \neq 0$, for all $i \leqslant j \leqslant n$.

Consequences:

- The characteristic varieties propagate: $\mathcal{V}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{V}^n(X, \mathbb{k})$.
- $\dim_{\mathbb{k}} H^1(X, \mathbb{k}) \geqslant n - 1$.
- If $n \geqslant 2$, then $H^i(X, \mathbb{k}) \neq 0$, for all $0 \leqslant i \leqslant n$.

THEOREM

If, moreover, X admits a minimal cell structure, then the resonance varieties also propagate: $\mathcal{R}^1(X, \mathbb{k}) \subseteq \cdots \subseteq \mathcal{R}^n(X, \mathbb{k})$.

HYPERPLANE ARRANGEMENTS

- Let $\mathcal{A}$ be a complex hyperplane arrangement, of rank n.
- Its complement, $M(\mathcal{A})$, has the homotopy type of a minimal CW-complex of dimension n.

THEOREM (DAVIS, JANUSZKIEWICZ, LEARY, OKUN 2011)

$M(\mathcal{A})$ is a duality space of dimension n.

THEOREM (DSY)

$M(\mathcal{A})$ is an abelian duality space of dimension n.

COROLLARY

The characteristic and resonance varieties of $M(\mathcal{A})$ propagate.

TORIC COMPLEXES

- Let L be simplicial complex on n vertices.
- The *toric complex* T_L is the subcomplex of the n-torus obtained by deleting the cells corresponding to the missing simplices of L.
- By construction, T_L is a minimal CW-complex, of dimension $\dim L + 1$.
- $\pi_\Gamma := \pi_1(T_L)$ is the *right-angled Artin group* associated to the graph $\Gamma = L^{(1)}$.
- $K(\pi_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
- $H^*(T_L, \Bbbk) = E/J_L$ is the *exterior Stanley–Reisner ring* of L.

- L is *Cohen–Macaulay* if for each simplex $\sigma \in L$, the reduced cohomology of $\operatorname{lk}(\sigma)$ is concentrated in degree $\dim(L) - |\sigma|$ and is torsion-free.

THEOREM (N. BRADY–MEIER 2001, JENSEN–MEIER 2005)

A right-angled Artin group π_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, π_Γ is a Poincaré duality group if and only if Γ is a complete graph.

THEOREM (DSY)

T_L is an abelian duality space (of dimension $\dim(L) + 1$) if and only if L is Cohen-Macaulay.

References

Graham Denham, Alexander I. Suciu, and Sergey Yuzvinsky, *Combinatorial covers and vanishing cohomology*, preprint, 2013.

Graham Denham, Alexander I. Suciu, and Sergey Yuzvinsky, *Abelian duality and propagation of resonance*, preprint, 2013.