Poincaré duality algebras and resonance varieties

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Resonance varieties

- Let $A^\bullet$ be a graded, graded-commutative, algebra (cga) over a field $\Bbbk$ with $\operatorname{char} \Bbbk \neq 2$, with multiplication maps $A^i \otimes_{\Bbbk} A^j \to A^{i+j}$.
- We assume A is connected ($A^0 = \Bbbk$) and of finite-type ($\dim_{\Bbbk} A^i < \infty$).
- For each $a \in A^1$ we have $a^2 = -a^2$, and so $a^2 = 0$.
- We then have a cochain complex,

$$(A^\bullet, \delta_a)\colon \ A^0 \xrightarrow{\delta_a^0} A^1 \xrightarrow{\delta_a^1} A^2 \xrightarrow{\delta_a^2} \cdots ,$$

  with differentials $\delta_a^i(u) = a \cdot u$, for all $u \in A^i$.
- The *resonance varieties* of A (in degree $i \geqslant 0$ and depth $k \geqslant 0$):

$$\mathcal{R}^i_k(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A^\bullet, \delta_a) \geqslant k\}.$$

- These sets are homogeneous subvarieties of the affine space A^1. For each $i \geqslant 0$, we have a descending filtration,

$$A^1 = \mathcal{R}^i_0(A) \supseteq \mathcal{R}^i_1(A) \supseteq \mathcal{R}^i_2(A) \cdots .$$

- An element $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if there exist $u_1, \dots, u_k \in A^i$ such that $au_1 = \cdots = au_k = 0$ in A^{i+1}, and the set $\{au, u_1, \dots, u_k\}$ is linearly independent in A^i, for all $u \in A^{i-1}$.
- If $\mathbb{k} \subset \mathbb{K}$ is a field extension, then the $\mathbb{k}$-points on $\mathcal{R}^i_k(A \otimes_{\mathbb{k}} \mathbb{K})$ coincide with $\mathcal{R}^i_k(A)$.
- Let $\varphi\colon A \to B$ be a morphism of cgas. If the map $\varphi^1\colon A^1 \to B^1$ is injective, then $\varphi^1(\mathcal{R}^1_k(A)) \subseteq \mathcal{R}^1_k(B)$, for all k.
- A linear subspace $U \subset A^1$ is *isotropic* if the restriction of $A^1 \wedge A^1 \to A^2$ to $U \wedge U$ is the zero map (i.e., $ab = 0$, $\forall a, b \in U$).
- If $U \subseteq A^1$ is an isotropic subspace of dimension k, then $U \subseteq \mathcal{R}^1_{k-1}(A)$.
- $\mathcal{R}^1_1(A)$ is the union of all isotropic planes in A^1.
- Let $W = \ker(A^1 \wedge A^1 \to A^2)$ and let $\mathrm{Gr}_2(A^1) \hookrightarrow \mathbb{P}(A^1 \wedge A^1)$ be the Plücker embedding. Then,

$$\mathcal{R}^1_1(A) = 0 \Longleftrightarrow \mathbb{P}(W) \cap \mathrm{Gr}_2(A^1) = \emptyset.$$

- Fix a $\Bbbk$-basis $\{e_1, \dots, e_n\}$ for A^1, let $\{x_1, \dots, x_n\}$ be the dual basis for $A_1 = (A^1)^*$, and identify $\mathrm{Sym}(A_1)$ with $S = \Bbbk[x_1, \dots, x_n]$, the coordinate ring of the affine space A^1.

- The BGG correspondence yields a cochain complex of finitely generated, free S-modules, $\mathrm{L}(A) := (A^\bullet \otimes_\Bbbk S, \delta)$,

$$\cdots \longrightarrow A^i \otimes_\Bbbk S \xrightarrow{\delta^i_A} A^{i+1} \otimes_\Bbbk S \xrightarrow{\delta^{i+1}_A} A^{i+2} \otimes_\Bbbk S \longrightarrow \cdots,$$

where $\delta^i_A(u \otimes s) = \sum_{j=1}^n e_j u \otimes s x_j$.

- The specialization of $(A \otimes_\Bbbk S, \delta)$ at $a \in A^1$ coincides with (A, δ_a).

- By definition, $a \in A^1$ belongs to $\mathcal{R}^i_k(A)$ if and only if $\operatorname{rank} \delta^{i-1}_a + \operatorname{rank} \delta^i_a \leqslant b_i(A) - k$. Hence,

$$\mathcal{R}^i_k(A) = V\Big(I_{b_i(A)-k+1}\big(\delta^{i-1}_A \oplus \delta^i_A\big)\Big).$$

- In particular, $\mathcal{R}^1_k(A) = V(I_{n-k}(\delta^1_A))$ $(0 \leqslant k < n)$ and $\mathcal{R}^1_n(A) = \{0\}$.

- The (degree i, depth k) *resonance scheme* $\boldsymbol{\mathcal{R}}^i_k(A)$ is defined by the determinantal ideal $I_{b_i(A)-k+1}\big(\delta^{i-1}_A \oplus \delta^i_A\big)$.

Examples

- If $E = \bigwedge \mathbb{k}^n$, then $\mathbf{L}(E)$ is the usual Koszul complex. E.g., for $n = 3$:

$$\mathbf{L}(E): \quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}} S \,.$$

Hence, $\mathcal{R}^i_k(E) = \{0\}$ for $0 \leqslant k \leqslant \binom{n}{i}$ and empty otherwise.

- If $A = \bigwedge(e_1, e_2, e_3)/\langle e_1 e_2 \rangle$, then

$$\mathbf{L}(A): \quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}} S^3 \xrightarrow{\begin{pmatrix} x_3 & 0 \\ 0 & x_3 \\ -x_1 & -x_2 \end{pmatrix}} S^2 \,.$$

Hence, $\mathcal{R}^1_1(A) = \{x_3 = 0\}$.

- If $A = \bigwedge(e_1, \dots, e_4)/\langle e_1 e_3, e_2 e_4, e_1 e_2 + e_3 e_4 \rangle$, then

$$\mathbf{L}(A): \quad S \xrightarrow{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} x_4 & 0 & -x_2 \\ 0 & x_3 & x_1 \\ 0 & -x_2 & x_4 \\ -x_1 & 0 & -x_3 \end{pmatrix}} S^3 \,.$$

Hence, $\mathcal{R}^1_1(A) = \{x_1 x_2 + x_3 x_4 = 0\}$.

Resonance varieties of spaces and groups

- Let X be a connected, finite-type CW-complex. The resonance varieties of X (over a field $\Bbbk$ with $\operatorname{char}\Bbbk \neq 2$) are the resonance varieties of its cohomology algebra: $\mathcal{R}^i_k(X,\Bbbk) := \mathcal{R}^i_k(H^\bullet(X,\Bbbk))$.
- The varieties $\mathcal{R}^1_k(X,\Bbbk)$ depend only on $G = \pi_1(X)$.
- The geometry of these varieties provides obstructions to the formality of X (or 1-formality of G).
- They allow to distinguish between various classes of groups, such as
    - Kähler groups
    - Quasi-projective groups
    - Arrangement groups
    - 3-manifold groups
    - Right-angled Artin groups
- Through their connections with other types of cohomology jump loci (characteristic varieties, Bieri–Neumann–Strebel–Renz invariants), they also inform on the homological and geometric finiteness properties of spaces and groups.

Poincaré duality algebras

- Let A be a connected, finite-type $\Bbbk$-cga.
- A is a *Poincaré duality* $\Bbbk$*-algebra* of dimension m if there is a $\Bbbk$-linear map $\varepsilon\colon A^m \to \Bbbk$ (called an *orientation*) such that all the bilinear forms $A^i \otimes_{\Bbbk} A^{m-i} \to \Bbbk$, $a \otimes b \mapsto \varepsilon(ab)$ are non-singular.
- We then have:
  - $b_i(A) = b_{m-i}(A)$, and $A^i = 0$ for $i > m$.
  - ε is an isomorphism.
  - The maps $\mathrm{PD}\colon A^i \to (A^{m-i})^*$, $\mathrm{PD}(a)(b) = \varepsilon(ab)$ are isos.
- Each $a \in A^i$ has a *Poincaré dual*, $a^\vee \in A^{m-i}$, such that $\varepsilon(aa^\vee) = 1$.
- The *orientation class* is $\omega_A := 1^\vee$.
- We have $\varepsilon(\omega_A) = 1$, and thus $aa^\vee = \omega_A$.

The associated alternating form

- Associated to a $\Bbbk$-PD$_m$ algebra there is an alternating m-form,

$$\mu_A\colon \bigwedge^m A^1 \to \Bbbk, \quad \mu_A(a_1 \wedge \cdots \wedge a_m) = \varepsilon(a_1 \cdots a_m).$$

- Assume now that $m = 3$, and set $n = b_1(A)$. Fix a basis $\{e_1, \dots, e_n\}$ for A^1, and let $\{e_1^\vee, \dots, e_n^\vee\}$ be the dual basis for A^2.

- The multiplication in A, then, is given on basis elements by

$$e_i e_j = \sum_{k=1}^{r} \mu_{ijk}\, e_k^\vee, \quad e_i e_j^\vee = \delta_{ij}\omega,$$

where $\mu_{ijk} = \mu(e_i \wedge e_j \wedge e_k)$.

- Let $A_i = (A^i)^*$. We may view μ dually as a trivector,

$$\mu = \sum \mu_{ijk}\, e^i \wedge e^j \wedge e^k \in \bigwedge^3 A_1,$$

which encodes the algebra structure of A.

Classification of alternating forms

- Let V be a $\mathbb{k}$-vector space of dimension n. The group $\mathrm{GL}(V)$ acts on $\bigwedge^m(V^*)$ by $(g \cdot \mu)(a_1 \wedge \cdots \wedge a_m) = \mu\left(g^{-1}a_1 \wedge \cdots \wedge g^{-1}a_m\right)$.
- The orbits of this action are the equivalence classes of alternating m-forms on V. (We write $\mu \sim \mu'$ if $\mu' = g \cdot \mu$.)
- Over $\overline{\mathbb{k}}$, the closures of these orbits are affine algebraic varieties; there are finitely many orbits only if $m \leqslant 2$ or $m = 3$ and $n \leqslant 8$.
- Each complex orbit has only finitely many real forms.
- When $m = 3$ and $n = 8$, there are 23 complex orbits, which split into either 1, 2, or 3 real orbits, for a total of 35 real orbits.

- Two PD_m algebras, A and B, are isomorphic as PD_m algebras if and only if they are isomorphic as graded algebras, in which case $\mu_A \sim \mu_B$.

Proposition

For two PD_3 algebras A and B, the following are equivalent.

(1) $A \cong B$, *as PD_3 algebras.*

(2) $A \cong B$, *as graded algebras.*

(3) $\mu_A \sim \mu_B$.

- We thus have a bijection between isomorphism classes of 3-dimensional Poincaré duality algebras and equivalence classes of alternating 3-forms, given by $A \leftrightsquigarrow \mu_A$.

Poincaré duality in orientable manifolds

- Let M be a compact, connected, orientable, m-dimensional manifold. Then the cohomology ring $A = H^{\bullet}(M, \Bbbk)$ is a PD_m algebra over $\Bbbk$.
- Sullivan (1975): for every finite-dimensional $\mathbb{Q}$-vector space V and every alternating 3-form $\mu \in \bigwedge^3 V^*$, there is a closed 3-manifold M with $H^1(M, \mathbb{Q}) = V$ and cup-product form $\mu_M = \mu$.
- Such a 3-manifold can be constructed via "Borromean surgery."
- E.g., 0-surgery on the Borromean rings in S^3 yields $M = T^3$, with $\mu_M = e^1 e^2 e^3$.
- If $M = \Sigma_g \times S^1$, where $g \geqslant 2$, then $\mu_M = \sum_{i=1}^{g} e^i e^{i+g} e^{2g+1}$.

Resonance varieties of PD-algebras

- Let A be a PD_m algebra. For $0 \leqslant i \leqslant m$ and $a \in A^1$, the following diagram commutes up to a sign.

$$\begin{array}{ccc}
(A^{m-i})^* & \xrightarrow{(\delta_{-a}^{m-i-1})^*} & (A^{m-i-1})^* \\
\mathrm{PD} \uparrow \cong & & \mathrm{PD} \uparrow \cong \\
A^i & \xrightarrow{\delta_a^i} & A^{i+1}
\end{array}$$

- Consequently, $\big(H^i(A, \delta_a)\big)^* \cong H^{m-i}(A, \delta_{-a})$.

- Hence, $\mathcal{R}^i_k(A) = \mathcal{R}^{m-i}_k(A)$ for all i and k. In particular, $\mathcal{R}^m_1(A) = \mathcal{R}^0_1(A) = \{0\}$.

Corollary

Let A be a PD_3 algebra with $b_1(A) = n$. Then $\mathcal{R}^i_k(A) = \emptyset$, except for:

- $\mathcal{R}^i_0(A) = A^1$ *for all* $i \geqslant 0$.
- $\mathcal{R}^3_1(A) = \mathcal{R}^0_1(A) = \{0\}$ *and* $\mathcal{R}^2_n(A) = \mathcal{R}^1_n(A) = \{0\}$.
- $\mathcal{R}^2_k(A) = \mathcal{R}^1_k(A)$ *for* $0 < k < n$.

- A linear subspace $U \subset V$ is 2-*singular* with respect to a 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ if $\mu(a \wedge b \wedge c) = 0$ for all $a, b \in U$ and $c \in V$.
- The *rank* of $\mu\colon \bigwedge^3 V \to \Bbbk$ is the minimum dimension of a linear subspace $W \subset V$ such that μ factors through $\bigwedge^3 W$. The *nullity* of μ is the maximum dimension of a 2-singular subspace $U \subset V$.
- Clearly, V contains a singular plane if and only if $\operatorname{null}(\mu) \geqslant 2$.
- Let A be a PD_3 algebra. A linear subspace $U \subset A^1$ is 2-singular (with respect to μ_A) if and only if U is isotropic.
- Using a result of A. Sikora [2005], we obtain:

Theorem

Let A be a PD_3 algebra over an algebraically closed field $\Bbbk$ with $\operatorname{char}(\Bbbk) \neq 2$, and let $\nu = \operatorname{null}(\mu_A)$. If $b_1(A) \geqslant 4$, then

$$\dim \mathcal{R}^1_{\nu-1}(A) \geqslant \nu \geqslant 2.$$

In particular, $\dim \mathcal{R}^1_1(A) \geqslant \nu$.

Real forms and resonance

- Sikora made the following conjecture: If $\mu\colon \bigwedge^3 V \to \Bbbk$ is a 3-form with $\dim V \geqslant 4$ and if $\operatorname{char}(\Bbbk) \neq 2$, then $\operatorname{null}(\mu) \geqslant 2$.
- Conjecture holds if $n := \dim V$ is even or equal to 5, or if $\Bbbk = \overline{\Bbbk}$.
- Work of J. Draisma and R. Shaw [2010, 2014] implies that the conjecture does not hold for $\Bbbk = \mathbb{R}$ and $n = 7$. We obtain:

Theorem

Let A be a PD_3 algebra over $\mathbb{R}$. Then $\mathcal{R}^1_1(A) \neq \{0\}$, except when

- $n = 1$, $\mu_A = 0$.
- $n = 3$, $\mu_A = e^1e^2e^3$.
- $n = 7$, $\mu_A = -e^1e^3e^5 + e^1e^4e^6 + e^2e^3e^6 + e^2e^4e^5 + e^1e^2e^7 + e^3e^4e^7 + e^5e^6e^7$.

Sketch: If $\mathcal{R}^1_1(A) = \{0\}$, then the formula $(x \times y) \cdot z = \mu_A(x, y, z)$ defines a cross-product on $A^1 = \mathbb{R}^n$, and thus a division algebra structure on $\mathbb{R}^{n+1}$, forcing $n = 1, 3$ or 7 by Bott–Milnor/Kervaire [1958].

Example

- Let A be the real PD_3 algebra corresponding to octonionic multiplication (the case $n = 7$ above).

- Let A' be the real PD_3 algebra with $\mu_{A'} = e^1e^2e^3 + e^4e^5e^6 + e^1e^4e^7 + e^2e^5e^7 + e^3e^6e^7$.

- Then $\mu_A \sim \mu_{A'}$ over $\mathbb{C}$, and so $A \otimes_{\mathbb{R}} \mathbb{C} \cong A' \otimes_{\mathbb{R}} \mathbb{C}$.

- On the other hand, $A \not\cong A'$ over $\mathbb{R}$, since $\mu_A \not\sim \mu_{A'}$ over $\mathbb{R}$, but also because $\mathcal{R}^1_1(A) = \{0\}$, yet $\mathcal{R}^1_1(A') \neq \{0\}$.

- Both $\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C})$ and $\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C})$ are projectively smooth conics, and thus are projectively equivalent over $\mathbb{C}$, but

$$\mathcal{R}^1_1(A \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1^2 + \cdots + x_7^2 = 0\}$$

has only one real point ($x = 0$), whereas

$$\mathcal{R}^1_1(A' \otimes_{\mathbb{R}} \mathbb{C}) = \{x \in \mathbb{C}^7 \mid x_1x_4 + x_2x_5 + x_3x_6 = x_7^2\}$$

contains the real (isotropic) subspace $\{x_4 = x_5 = x_6 = x_7 = 0\}$.

Pfaffians and resonance

Let A be a $\Bbbk$-PD$_3$ algebra with $b_1(A) = n$. The cochain complex $\mathsf{L}(A) = (A \otimes_{\Bbbk} S, \delta_A)$ then looks like

$$A^0 \otimes_{\Bbbk} S \xrightarrow{\delta_A^0} A^1 \otimes_{\Bbbk} S \xrightarrow{\delta_A^1} A^2 \otimes_{\Bbbk} S \xrightarrow{\delta_A^2} A^3 \otimes_{\Bbbk} S\,,$$

where $\delta_A^0 = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}$ and $\delta_A^2 = (\delta_A^0)^\top$, while δ_A^1 is the skew- symmetric matrix whose are entries linear forms in S given by

$$\delta_A^1(e_i) = \sum\nolimits_{j=1}^n \sum\nolimits_{k=1}^n \mu_{jik}\, e_k^\vee \otimes x_j\,.$$

Theorem

We have $\mathcal{R}^1_{2k}(A) = \mathcal{R}^1_{2k+1}(A) = V(\mathrm{Pf}_{n-2k}(\delta_A^1))$ *if* n *is even and* $\mathcal{R}^1_{2k-1}(A) = \mathcal{R}^1_{2k}(A) = V(\mathrm{Pf}_{n-2k+1}(\delta_A^1))$ *if* n *is odd. Moreover, if* μ_A *has maximal rank* $n \geqslant 3$*, then*

$$\mathcal{R}^1_{n-2}(A) = \mathcal{R}^1_{n-1}(A) = \mathcal{R}^1_n(A) = \{0\}.$$

Resonance varieties of 3-forms of low rank

$\mathbb{C}$	μ	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$
I	0	$\emptyset$	$\emptyset$	$\emptyset$
II	123	0	0	0
III	125 + 345	$\{x_5 = 0\}$	$\{x_5 = 0\}$	0

$\mathbb{C}$	$\mathbb{R}$	μ	$\mathcal{R}_1$	$\mathcal{R}_2 = \mathcal{R}_3$	$\mathcal{R}_4$
IV		135 + 234 + 126	$\Bbbk^6$	$\{x_1 = x_2 = x_3 = 0\}$	0
V	a	123 + 456	$\Bbbk^6$	$\{x_1 = x_2 = x_3 = 0\} \cup \{x_4 = x_5 = x_6 = 0\}$	0
	b	−135 + 146 + 236 + 245	$\Bbbk^6$	$V(x_1^2 + x_2^2, x_3^2 + x_4^2, x_5^2 + x_6^2, x_4x_5 - x_3x_6, x_3x_5 + x_4x_6, x_2x_5 - x_1x_6, x_1x_5 + x_2x_6, x_2x_3 - x_1x_4, x_1x_3 + x_2x_4)$	0

$\mathbb{C}$	$\mathbb{R}$	μ	$\mathcal{R}_1 = \mathcal{R}_2$	$\mathcal{R}_3 = \mathcal{R}_4$
VI		123 + 145 + 167	$\{x_1 = 0\}$	$\{x_1 = 0\}$
VII		125 + 136 + 147 + 234	$\{x_1 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\}$
VIII	a	134 + 256 + 127	$\{x_1 = 0\} \cup \{x_2 = 0\}$	$\{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_2 = x_5 = x_6 = 0\}$
	b	−135 + 146 + 236 + 245 + 127	$\{x_1^2 + x_2^2 = 0\}$	$V(x_1, x_2, x_3^2 + x_4^2, x_5^2 + x_6^2, x_3x_5 + x_4x_6, x_4x_5 - x_3x_6)$
IX	a	125 + 346 + 137 + 247	$\{x_1x_4 + x_2x_5 = 0\}$	$V(x_7^2 - x_3x_6, x_1, x_2, x_4, x_5)$
	b	−135 + 146 + 236 + 245 + 127 + 347	$\{x_1x_3 + x_2x_4 = 0\}$	$V(x_7^2 - x_5x_6, x_1, x_2, x_3, x_4)$
X	a	123 + 456 + 147 + 257 + 367	$\{x_1x_4 + x_2x_5 + x_3x_6 = x_7^2\}$	0
	b	−135 + 146 + 236 + 245 + 127 + 347 + 567	$\{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 = 0\}$	0

Lemma (Turaev 2002)

Suppose $n \geqslant 3$. There is then a polynomial $\operatorname{Det}(\mu_A) \in \operatorname{Sym}(A_1)$ such that, if $\delta^1_A(i;j)$ is the sub-matrix obtained from δ^1_A by deleting the i-th row and j-th column, then $\det \delta^1_A(i;j) = (-1)^{i+j} x_i x_j \operatorname{Det}(\mu_A)$.

Moreover, if n is even, then $\operatorname{Det}(\mu_A) = 0$, while if n is odd, then $\operatorname{Det}(\mu_A) = \operatorname{Pf}(\mu_A)^2$, where $\operatorname{pf}(\delta^1_A(i;i)) = (-1)^{i+1} x_i \operatorname{Pf}(\mu_A)$.

- Suppose $\dim V = 2g+1 > 1$. A 3-form $\mu\colon \bigwedge^3 V \to \Bbbk$ is *generic* (in the sense of Berceanu–Papadima [1994]) if there is a $v \in V$ such that the 2-form $\gamma_v \in V^* \wedge V^*$ given by $\gamma_v(a \wedge b) = \mu_A(a \wedge b \wedge v)$ for $a, b \in V$ has rank $2g$, that is, $\gamma_v^g \neq 0$ in $\bigwedge^{2g} V^*$.

Example

Let $M = \Sigma_g \times S^1$, where $g \geqslant 2$. Then $\mu_M = \sum_{i=1}^{g} e^i e^{i+1} e^{2g+1}$ is BP-generic, and $\operatorname{Pf}(\mu_M) = x_{2g+1}^{g-1}$. Hence, $\mathcal{R}^1_1(M) = \{x_{2g+1} = 0\}$. In fact,

$$\mathcal{R}^1_1 = \cdots = \mathcal{R}^1_{2g-2} = \{x_{2g+1} = 0\} \text{ and } \mathcal{R}^1_{2g-1} = \mathcal{R}^1_{2g} = \mathcal{R}^1_{2g+1} = \{0\}.$$

Lemma

If n is odd and $n > 1$, then $\mathcal{R}^1_1(A) \neq A^1 \iff \mu_A$ is BP-generic.

Theorem

Let A be a PD_3 algebra with $b_1(A) = n$. Then

$$\mathcal{R}^1_1(A) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{0\} & \text{if } n = 1 \text{ or } n = 3 \text{ and } \mu \text{ has rank } 3 \\ V(\mathrm{Pf}(\mu_A)) & \text{if } n \text{ is odd, } n > 3, \text{ and } \mu_A \text{ is BP-generic} \\ A^1 & \text{otherwise.} \end{cases}$$

- If M is a closed orientable 3-manifold with $b_1(M)$ even and positive, the equality $\mathcal{R}^1_1(M) = H^1(M, \mathbb{C})$ was first proved in [Dimca–S. 2009].
- We used this to show that the only 3-manifold groups which are also Kähler groups are the finite subgroups of $\mathrm{O}(4)$.
- Moreover, if M fibers over the circle, then M is not 1-formal [Papadima–S. 2010].

As a corollary, we recover a closely related result, proved by Draisma and Shaw [2010] by very different methods.

Corollary

Let V be a $\Bbbk$-vector space of odd dimension $n \geqslant 5$ and let $\mu \in \bigwedge^3 V^\vee$. Then the union of all singular planes is either all of V or a hypersurface defined by a homogeneous polynomial in $\Bbbk[V]$ of degree $(n-3)/2$.

For $\mu \in \bigwedge^3 V^\vee$, there is another genericity condition, due to P. De Poi, D. Faenzi, E. Mezzetti, and K. Ranestad [2017]: $\operatorname{rank}(\gamma_v) > 2$, for all non-zero $v \in V$. We may interpret some of their results, as follows.

Theorem (DFMR)

Let A be a PD_3 algebra over $\mathbb{C}$, and suppose μ_A is generic. Then:

- *If n is odd, then $\mathcal{R}^1_1(A)$ is a hypersurface of degree $(n-3)/2$ which is smooth if $n \leqslant 7$, and singular in codimension 5 if $n \geqslant 9$.*
- *If n is even, then $\mathcal{R}^1_2(A)$ has codim 3 and degree $\frac{1}{4}\binom{n-2}{3}+1$; it is smooth if $n \leqslant 10$, and singular in codimension 7 if $n \geqslant 12$.*

Bockstein resonance varieties

- Let X be a connected, finite-type CW-complex and let $A = H^{\bullet}(X, \mathbb{Z}_2)$.
- For each $q \geqslant 0$, we have a *Bockstein operator*,

$$\beta_2 \colon H^q(X, \mathbb{Z}_2) \to H^{q+1}(X, \mathbb{Z}_2),$$

defined as the coboundary homomorphism associated to the coefficient exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\times 2} \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

- $\beta_2 \colon A^{\bullet} \to A^{\bullet+1}$ is a differential and $\beta_2(a) = a^2$ for all $a \in A^1$.
- For each $a \in A^1$, we obtain a cochain complex of finite-dimensional $\mathbb{Z}_2$-vector spaces,

$$(A, \delta_a) \colon \; A^0 \xrightarrow{\delta_a} A^1 \xrightarrow{\delta_a} \cdots \xrightarrow{\delta_a} A^i \xrightarrow{\delta_a} A^{i+1} \xrightarrow{\delta_a} \cdots,$$

where $\delta_a(u) = au + \beta_2(u)$.

- Pick basis $\{e_1, \dots, e_n\}$ for $A^1 = H^1(X, \mathbb{Z}_2)$, let $\{x_1, \dots, x_n\}$ be dual basis for $A_1 = H_1(X, \mathbb{Z}_2)$, and identify $\mathrm{Sym}(A_1) \cong \mathbb{Z}_2[x_1, \dots, x_n]$.
- The coordinate ring of A^1 is then

$$S = \mathbb{Z}_2[x_1, \dots, x_n]/(x_1^2 + x_1, \dots, x_n^2 + x_n).$$

  This is the ring of (Boolean) functions on $\mathbb{Z}_2^n$.
- We then have a cochain complex of free S-modules,

$$(A \otimes_{\mathbb{Z}_2} S, \delta)\colon\ A^0 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^0} A^1 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^1} A^2 \otimes_{\mathbb{Z}_2} S \xrightarrow{\delta^2} \cdots,$$

  where $\delta^i(u \otimes 1) = \sum_{j=1}^n e_j u \otimes x_j + \beta_2(u) \otimes 1$ for $u \in A^i$, whose specialization at $a \in A^1$ is (A, δ_a).
- We define the *Bockstein resonance varieties* of X as

$$\widetilde{\mathcal{R}}^q_k(X, \mathbb{Z}_2) = \big\{a \in H^1(X, \mathbb{Z}_2) \mid \dim_{\mathbb{Z}_2} H^q(A, \delta_a) \geqslant k\big\}.$$

- More generally, if $\mathrm{char}(\Bbbk) = 2$, then $\widetilde{\mathcal{R}}^q_k(X, \Bbbk) = \widetilde{\mathcal{R}}^q_k(X, \mathbb{Z}_2) \times_{\mathbb{Z}_2} \Bbbk$.

Poincaré duality and resonance over $\mathbb{Z}_2$

- If $H_1(X,\mathbb{Z})$ has no 2-torsion, then $\mathcal{R}^1_k(X,\mathbb{Z}_2)=\widetilde{\mathcal{R}}^1_k(X,\mathbb{Z}_2)$, $\forall k$.
- $\mathcal{R}^q_k(X,\mathbb{Z}_2)\neq\widetilde{\mathcal{R}}^q_k(X,\mathbb{Z}_2)$ for $q>1$ (neither inclusion needs to hold).

Theorem

Let M be a closed m-manifold. The following are equivalent:

(1) M is orientable

(2) $\beta_2\colon H^{m-1}(M,\mathbb{Z}_2)\to H^m(M,\mathbb{Z}_2)$ is zero.

(3) $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)=\{0\}$.

Proposition

Let M be a closed, orientable m-manifold, and assume $\operatorname{char}(\Bbbk)=2$. Then $\widetilde{\mathcal{R}}^i_k(M;\Bbbk)=\widetilde{\mathcal{R}}^{m-i}_k(M;\Bbbk)$ for all i,k. In particular, $\widetilde{\mathcal{R}}^m_1(M,\Bbbk)=\{0\}$.

Proposition

Let M be a closed, non-orientable m-manifold such that $H_1(M,\mathbb{Z})$ has no 2-torsion. Then $\mathcal{R}^m_1(M,\mathbb{Z}_2)=\{0\}$ whereas $\widetilde{\mathcal{R}}^m_1(M,\mathbb{Z}_2)=\mathbb{Z}_2$.

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