Partial products of circles

Alex Suciu

Northeastern University
Boston, Massachusetts

Algebra and Geometry Seminar
Vrije University Amsterdam
October 13, 2009

1. Toric complexes
   - Partial products of spaces
   - Toric complexes and right-angled Artin groups
   - Graded Lie algebras associated to RAAGs
   - Chen Lie algebras of RAAGs
   - Artin kernels and Bestvina-Brady groups

2. Resonance varieties
   - Resonance varieties
   - Kähler and quasi-Kähler groups
   - Kähler and quasi-Kähler RAAGs
   - Kähler and quasi-Kähler BB groups

Partial product construction

Input:

- K, a simplicial complex on $[n] = \{1, \dots, n\}$.
- (X, A), a pair of topological spaces, $A \neq \emptyset$.

Output:

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma \subset X^{\times n}$$

where $(X, A)^\sigma = \{x \in X^{\times n} \mid x_i \in A \text{ if } i \notin \sigma\}$.
Interpolates between

- $\mathcal{Z}_\emptyset(X, A) = \mathcal{Z}_K(A, A) = A^{\times n}$ and
- $\mathcal{Z}_{\Delta^{n-1}}(X, A) = \mathcal{Z}_K(X, X) = X^{\times n}$

Examples:

- $\mathcal{Z}_{n \text{ points}}(X, *) = \bigvee^n X$ (wedge)
- $\mathcal{Z}_{\partial\Delta^{n-1}}(X, *) = T^n X$ (fat wedge)

Properties:

- $L \subset K$ subcomplex $\Rightarrow \mathcal{Z}_L(X,A) \subset \mathcal{Z}_K(X,A)$ subspace.
- (X,A) pair of (finite) CW-complexes $\Rightarrow \mathcal{Z}_K(X,A)$ is a (finite) CW-complex.
- $\mathcal{Z}_{K*L}(X,A) \cong \mathcal{Z}_K(X,A) \times \mathcal{Z}_L(X,A)$.
- $f\colon (X,A) \to (Y,B)$ continuous map $\Rightarrow f^{\times n}\colon X^{\times n} \to Y^{\times n}$ restricts to a continuous map $\mathcal{Z}^f\colon \mathcal{Z}_K(X,A) \to \mathcal{Z}_K(Y,B)$.
- Consequently, $(X,A) \simeq (Y,B) \Rightarrow \mathcal{Z}_K(X,A) \simeq \mathcal{Z}_K(Y,B)$.
- (Strickland) $f\colon K \to L$ simplicial $\rightsquigarrow \mathcal{Z}_f\colon \mathcal{Z}_K(X,A) \to \mathcal{Z}_L(X,A)$ continuous (if X connected topological monoid, A submonoid).
- (Denham–S. 2005) If $(M,\partial M)$ is a compact manifold of dim d, and K is a PL-triangulation of S^m on n vertices, then $\mathcal{Z}_K(M,\partial M)$ is a compact manifold of dim $(d-1)n+m+1$.
- (Bosio–Meersseman 2006) If K is a polytopal triangulation of S^m, then $\mathcal{Z}_K(D^2,S^1)$ if $n+m+1$ is even, or $\mathcal{Z}_K(D^2,S^1)\times S^1$ if $n+m+1$ is odd, is a complex manifold.

Toric complexes and right-angled Artin groups

Definition

Let L be simplicial complex on n vertices. The associated *toric complex*, T_L, is the subcomplex of the n-torus obtained by deleting the cells corresponding to the missing simplices of L, i.e.,

$$T_L = \mathcal{Z}_L(S^1, *).$$

- k-cells in $T_L \longleftrightarrow (k-1)$-simplices in L.
- $C_*^{\mathrm{CW}}(T_L)$ is a subcomplex of $C_*^{\mathrm{CW}}(T^n)$; thus, all $\partial_k = 0$, and

$$H_k(T_L, \mathbb{Z}) = C_{k-1}^{\text{simplicial}}(L, \mathbb{Z}) = \mathbb{Z}^{\#\,(k-1)\text{-simplices of } L}.$$

- $H^*(T_L, \Bbbk)$ is the *exterior Stanley-Reisner ring* $\bigwedge V^*/J_L$, where
  - V is the free $\Bbbk$-module on the vertex set of L;
  - $\bigwedge_{\Bbbk} V^*$ is the exterior algebra on dual of V;
  - J_L is the ideal generated by all monomials, $v_\sigma = v_{i_1}^* \cdots v_{i_k}^*$ corresponding to simplices $\sigma = \{v_{i_1}, \dots, v_{i_k}\}$ not belonging to L.

Right-angled Artin groups

Definition

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ be a (finite, simple) graph. The corresponding *right-angled Artin group* is

$$G_\Gamma = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle.$$

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n; \quad \Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}; \quad \Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$
- $\Gamma \cong \Gamma' \Leftrightarrow G_\Gamma \cong G_{\Gamma'}$
  (Kim–Makar-Limanov–Neggers–Roush 1980)
- $\pi_1(T_L) = G_\Gamma$, where $\Gamma = L^{(1)}$.
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
  (Davis–Charney 1995, Meier–VanWyk 1995)
- $A := H^*(G_\Gamma, \Bbbk) = \bigwedge_{\Bbbk} V^*/J_\Gamma$, where J_Γ is quadratic monomial ideal
  $\Rightarrow$ A is a Koszul algebra (Fröberg 1975).

Formality

Definition (Sullivan)

A space X is *formal* if its minimal model is quasi-isomorphic to $(H^*(X,\mathbb{Q}),0)$.

Definition (Quillen)

A group G is 1*-formal* if its Malcev Lie algebra, $\mathfrak{m}_G = \mathrm{Prim}(\widehat{\mathbb{Q}G})$, is a (complete, filtered) quadratic Lie algebra.

Theorem (Sullivan)

If X formal, then $\pi_1(X)$ is 1-formal.

Theorem (Notbohm–Ray 2005)

T_L is formal, and so G_Γ is 1-formal.

Associated graded Lie algebra

Let G be a finitely-generated group. Define:

- *LCS series*: $G = G_1 \triangleright G_2 \triangleright \cdots \triangleright G_k \triangleright \cdots$, where $G_{k+1} = [G_k, G]$
- *LCS quotients*: $\operatorname{gr}_k G = G_k / G_{k+1}$ (f.g. abelian groups)
- *LCS ranks*: $\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)$
- *Associated graded Lie algebra*: $\operatorname{gr}(G) = \bigoplus_{k \geq 1} \operatorname{gr}_k(G)$, with Lie bracket $[\,,\,]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by group commutator.

Example (Witt, Magnus)

Let $G = F_n$ (free group of rank n).
Then $\operatorname{gr} G = \operatorname{Lie}_n$ (free Lie algebra of rank n), with LCS ranks given by

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = 1 - nt.$$

Explicitly: $\phi_k(F_n) = \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}$, where μ is Möbius function.

Holonomy Lie algebra

Definition (Chen 1977, Markl–Papadima 1992)

Let G be a finitely generated group, with $H_1 = H_1(G, \mathbb{Z})$ torsion-free. The *holonomy Lie algebra* of G is the quadratic, graded Lie algebra

$$\mathfrak{h}_G = \mathrm{Lie}(H_1)/\mathrm{ideal}(\mathrm{im}(\nabla)),$$

where $\nabla\colon H_2(G,\mathbb{Z}) \to H_1 \wedge H_1 = \mathrm{Lie}_2(H_1)$ is the comultiplication map.

Let $G = \pi_1(X)$ and $A = H^*(X, \mathbb{Q})$.

- (Löfwall 1986) $U(\mathfrak{h}_G \otimes \mathbb{Q}) \cong \bigoplus_{k\geq 1} \mathrm{Ext}^k_A(\mathbb{Q},\mathbb{Q})_k$.
- There is a canonical epimorphism $\mathfrak{h}_G \twoheadrightarrow \mathrm{gr}(G)$.
- (Sullivan) If G is 1-formal, then $\mathfrak{h}_G \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$.

Example

$G = F_n$, then clearly $\mathfrak{h}_G = \mathrm{Lie}_n$, and so $\mathfrak{h}_G = \mathrm{gr}(G)$.

Let $\Gamma = (\mathsf{V}, \mathsf{E})$ graph, and $P_\Gamma(t) = \sum_{k\geq 0} f_k(\Gamma) t^k$ its clique polynomial.

Theorem (Duchamp–Krob 1992, Papadima–S. 2006)

For $G = G_\Gamma$:

1. $\operatorname{gr}(G) \cong \mathfrak{h}_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = P_\Gamma(-t).$$

Idea of proof:

1. $A = \bigwedge_{\Bbbk} \mathsf{V}^*/J_\Gamma \Rightarrow \mathfrak{h}_G \otimes \Bbbk = L_\Gamma := \operatorname{Lie}(\mathsf{V})/([v,w] = 0 \text{ if } \{v,w\} \in \mathsf{E})$.
2. Shelton–Yuzvinsky: $U(L_\Gamma) = A^!$ (Koszul dual).
3. Koszul duality: $\operatorname{Hilb}(A^!, t) \cdot \operatorname{Hilb}(A, -t) = 1$.
4. $\operatorname{Hilb}(\mathfrak{h}_G \otimes \Bbbk, t)$ independent of $\Bbbk \Rightarrow \mathfrak{h}_G$ torsion-free.
5. But $\mathfrak{h}_G \twoheadrightarrow \operatorname{gr}(G)$ is iso over $\mathbb{Q}$ (by 1-formality) $\Rightarrow$ iso over $\mathbb{Z}$.
6. LCS formula follows from (3) and PBW.

Chen Lie algebras

Definition (Chen 1951)

The *Chen Lie algebra* of a (finitely generated) group G is $\mathrm{gr}(G/G'')$, i.e., the assoc. graded Lie algebra of its maximal metabelian quotient. Write $\theta_k(G) = \operatorname{rank} \mathrm{gr}_k(G/G'')$ for the Chen ranks.

Facts:

- $\mathrm{gr}(G) \twoheadrightarrow \mathrm{gr}(G/G'')$, and so $\phi_k(G) \geq \theta_k(G)$, with equality for $k \leq 3$.
- The map $\mathfrak{h}_G \twoheadrightarrow \mathrm{gr}(G)$ induces epimorphism $\mathfrak{h}_G/\mathfrak{h}''_G \twoheadrightarrow \mathrm{gr}(G/G'')$.
- (P.–S. 2004) If G is 1-formal, then $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G/G'') \otimes \mathbb{Q}$.

Example (Chen)

$$\theta_k(F_n) = \binom{n+k-2}{k}(k-1), \quad \text{for all } k \geq 2.$$

The Chen Lie algebra of a RAAG

Theorem (P.–S. 2006)

For $G = G_\Gamma$:

1. $\operatorname{gr}(G/G'') \cong \mathfrak{h}_G/\mathfrak{h}''_G$.
2. *Graded pieces are torsion-free, with ranks given by*

$$\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\left(\frac{t}{1-t}\right),$$

where $Q_\Gamma(t) = \sum_{j\geq 2} c_j(\Gamma) t^j$ is the "cut polynomial" of Γ, with

$$c_j(\Gamma) = \sum_{\mathsf{W}\subset\mathsf{V}:\ |\mathsf{W}|=j} \tilde{b}_0(\Gamma_{\mathsf{W}}).$$

Idea of proof:

1. Write $A := H^*(G, \Bbbk) = E/J_\Gamma$, where $E = \bigwedge_{\Bbbk}(v_1^*, \dots, v_n^*)$.
2. Write $\mathfrak{h} = \mathfrak{h}_G \otimes \Bbbk$.
3. By Fröberg and Löfwall (2002)
$$\left(\mathfrak{h}'/\mathfrak{h}''\right)_k \cong \operatorname{Tor}^E_{k-1}(A, \Bbbk)_k, \quad \text{for } k \geq 2$$
4. By Aramova–Herzog–Hibi & Aramova–Avramov–Herzog (97-99):
$$\sum_{k\geq 2} \dim_{\Bbbk} \operatorname{Tor}^E_{k-1}(E/J_\Gamma, \Bbbk)_k = \sum_{i\geq 1} \dim_{\Bbbk} \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} \cdot \left(\frac{t}{1-t}\right)^{i+1},$$
where $S = \Bbbk[x_1, \dots, x_n]$ and $I_\Gamma = \text{ideal}\,\langle x_i x_j \mid \{v_i, v_j\} \notin \mathsf{E}\rangle$.
5. By Hochster (1977):
$$\dim_{\Bbbk} \operatorname{Tor}^S_i(S/I_\Gamma, \Bbbk)_{i+1} = \sum_{\mathsf{W}\subset \mathsf{V}:\ |\mathsf{W}|=i+1} \dim_{\Bbbk} \widetilde{H}_0(\Gamma_{\mathsf{W}}, \Bbbk) = c_{i+1}(\Gamma).$$
6. The answer is independent of $\Bbbk \Rightarrow \mathfrak{h}_G/\mathfrak{h}''_G$ is torsion-free.
7. Using formality of G_Γ, together with $\mathfrak{h}_G/\mathfrak{h}''_G \otimes \mathbb{Q} \xrightarrow{\simeq} \operatorname{gr}(G/G'') \otimes \mathbb{Q}$ ends the proof.

Example

Let Γ be a pentagon, and Γ' a square with an edge attached to a vertex. Then:

- $P_\Gamma = P_{\Gamma'} = 1 - 5t + 5t^2$, and so
$$\phi_k(G_\Gamma) = \phi_k(G_{\Gamma'}), \quad \text{for all } k \geq 1.$$
- $Q_\Gamma = 5t^2 + 5t^3$ but $Q_{\Gamma'} = 5t^2 + 5t^3 + t^4$, and so
$$\theta_k(G_\Gamma) \neq \theta_k(G_{\Gamma'}), \quad \text{for } k \geq 4.$$

Artin kernels

Definition

Given a graph Γ, and an epimorphism $\chi\colon G_\Gamma \twoheadrightarrow \mathbb{Z}$, the corresponding *Artin kernel* is the group

$$N_\chi = \ker(\chi\colon G_\Gamma \to \mathbb{Z})$$

Note that $N_\chi = \pi_1(T_L^\chi)$, where $T_L^\chi \to T_L$ is the regular $\mathbb{Z}$-cover defined by χ. A classifying space for N_χ is $T_{\Delta_\Gamma}^\chi$, where $\Gamma = L^{(1)}$.

Noteworthy is the case when χ is the "diagonal" homomorphism $\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z}$, which assigns to each vertex the value 1.

The corresponding Artin kernel, $N_\Gamma = N_\nu$, is called the *Bestvina–Brady group* associated to Γ.

Stallings, Bieri, Bestvina and Brady: geometric and homological finiteness properties of $N_\Gamma \longleftrightarrow$ topology of Δ_Γ, e.g.:

- N_Γ is finitely generated $\iff \Gamma$ is connected
- N_Γ is finitely presented $\iff \Delta_\Gamma$ is simply-connected.

More generally, it follows from Meier–Meinert–VanWyk (1998) and Bux–Gonzalez (1999) that:

Theorem

Assume L is a flag complex. Let $\mathsf{W} = \{v \in \mathsf{V} \mid \chi(v) \neq 0\}$ *be the support of* χ*. Then:*

1. N_χ *is finitely generated* $\iff L_{\mathsf{W}}$ *is connected, and,* $\forall\, v \in \mathsf{V} \setminus \mathsf{W}$*, there is a* $w \in \mathsf{W}$ *such that* $\{v, w\} \in L$*.*
2. N_χ *is finitely presented* $\iff L_{\mathsf{W}}$ *is* 1*-connected and,* $\forall\, \sigma \in L_{\mathsf{V} \setminus \mathsf{W}}$*, the space* $\operatorname{lk}_{L_{\mathsf{W}}}(\sigma) = \{\tau \in L_{\mathsf{W}} \mid \tau \cup \sigma \in L\}$ *is* $(1 - |\sigma|)$*-acyclic.*

Theorem (P.–S. 2009)

Let Γ be a graph, and N_χ and Artin kernel.

1. *If $H_1(N_\chi, \mathbb{Q})$ is a trivial $\mathbb{Q}\mathbb{Z}$-module, then N_χ is finitely generated.*
2. *If both $H_1(N_\chi, \mathbb{Q})$ and $H_2(N_\chi, \mathbb{Q})$ have trivial $\mathbb{Z}$-action, then N_χ is 1-formal.*

Thus, if Γ is connected, and $H_1(\Delta_\Gamma, \mathbb{Q}) = 0$, then N_Γ is 1-formal.

Theorem (P.–S. 2009)

Suppose $H_1(N, \mathbb{Q})$ has trivial $\mathbb{Z}$-action. Then, both $\operatorname{gr}(N)$ and $\operatorname{gr}(N/N'')$ are torsion-free, with graded ranks, ϕ_k and θ_k, given by

$$\prod_{k=1}^{\infty}(1 - t^k)^{\phi_k} = \frac{P_\Gamma(-t)}{1 - t},$$

$$\sum_{k=2}^{\infty}\theta_k t^k = Q_\Gamma\left(\frac{t}{1 - t}\right).$$

Resonance varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$).
Let $\Bbbk$ be a field; if $\operatorname{char} \Bbbk = 2$, assume $H_1(X, \mathbb{Z})$ has no 2-torsion.
Let $A = H^*(X, \Bbbk)$. Then: $a \in A^1 \Rightarrow a^2 = 0$. Thus, get cochain complex

$$(A, \cdot a)\colon\ A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

Definition (Falk 1997, Matei–S. 2000)

The *resonance varieties* of X (over $\Bbbk$) are the algebraic sets

$$\mathcal{R}^i_d(X, \Bbbk) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

- $\mathcal{R}^i_d$ are homogeneous subvarieties of $A^1 = H^1(X, \Bbbk)$
- $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots \supseteq \mathcal{R}^i_{b_i+1} = \emptyset$, where $b_i = b_i(X, \Bbbk)$.
- $\mathcal{R}^1_d(X, \Bbbk)$ depends only on $G = \pi_1(X)$, so denote it by $\mathcal{R}_d(G, \Bbbk)$.

Resonance of toric complexes

Recall $A = H^*(T_L, \Bbbk)$ is the exterior Stanley-Reisner ring of L. Using a formula of Aramova, Avramov, and Herzog (1999), we prove:

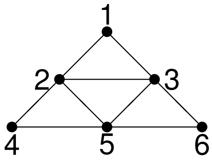
Theorem (P.–S. 2009)

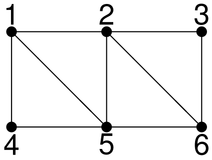
$$\mathcal{R}^i_d(T_L, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subset \mathsf{V} \\ \sum_{\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}} \dim_\Bbbk \widetilde{H}_{i-1-|\sigma|}(\mathrm{lk}_{L_\mathsf{W}}(\sigma), \Bbbk) \geq d}} \Bbbk^{\mathsf{W}},$$

where L_W is the subcomplex induced by L on W, and $\mathrm{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{R}^1_1(G_\Gamma, \Bbbk) = \bigcup_{\substack{\mathsf{W} \subseteq \mathsf{V} \\ \Gamma_\mathsf{W} \text{ disconnected}}} \Bbbk^{\mathsf{W}}.$$





Example

Let Γ and Γ' be the two graphs above. Both have

$$P(t) = 1 + 6t + 9t^2 + 4t^3, \quad \text{and} \quad Q(t) = t^2(6 + 8t + 3t^2).$$

Thus, G_Γ and $G_{\Gamma'}$ have the same LCS and Chen ranks.
Each resonance variety has 3 components, of codimension 2:

$$\mathcal{R}_1(G_\Gamma, \Bbbk) = \Bbbk^{\overline{23}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{35}}, \qquad \mathcal{R}_1(G_{\Gamma'}, \Bbbk) = \Bbbk^{\overline{15}} \cup \Bbbk^{\overline{25}} \cup \Bbbk^{\overline{26}}.$$

Yet the two varieties are not isomorphic, since

$$\dim(\Bbbk^{\overline{23}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{35}}) = 3, \quad \text{but} \quad \dim(\Bbbk^{\overline{15}} \cap \Bbbk^{\overline{25}} \cap \Bbbk^{\overline{26}}) = 2.$$

Kähler manifolds

Definition

A compact, connected, complex manifold M is called a *Kähler manifold* if M admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form.

Examples: Riemann surfaces, $\mathbb{CP}^n$, and, more generally, smooth, complex projective varieties.

Definition

A group G is a *Kähler group* if $G = \pi_1(M)$, for some compact Kähler manifold M.

G is *projective* if M is actually a smooth projective variety.

- G finite $\Rightarrow$ G is a projective group (Serre 1958).
- G_1, G_2 Kähler groups $\Rightarrow$ $G_1 \times G_2$ is a Kähler group
- G Kähler group, $H < G$ finite-index subgroup $\Rightarrow$ H is a Kähler gp

Problem (Serre 1958)

Which finitely presented groups are Kähler (or projective) groups?

The Kähler condition puts strong restrictions on M:

1. $H^*(M, \mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of M are even
3. M is formal, i.e., $(\Omega(M), d) \simeq (H^*(M, \mathbb{R}), 0)$ (Deligne–Griffiths–Morgan–Sullivan 1975)

The Kähler condition also puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. G is 1-formal, i.e., its Malcev Lie algebra $\mathfrak{m}(G)$ is quadratic
3. G cannot split non-trivially as a free product (Gromov 1989)

Quasi-Kähler manifolds

Definition

A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact Kähler manifold and D is a divisor with normal crossings.

Similar definition for X quasi-projective.
The notions of quasi-Kähler group and quasi-projective group are defined as above.

- X quasi-projective $\Rightarrow H^*(X, \mathbb{Z})$ has a mixed Hodge structure (Deligne 1972–74)
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal (Brieskorn 1973)
- X quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal (Kohno 1983)

Resonance varieties of quasi-Kähler manifolds

Theorem (D.–P.–S. 2009)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}_1(G)$. If G is 1-formal, then

1. *Each L_α is a p-isotropic linear subspace of $H^1(G, \mathbb{C})$, with* $\dim L_\alpha \geq 2p + 2$, *for some $p = p(\alpha) \in \{0, 1\}$.*
2. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
3. $\mathcal{R}_d(G) = \{0\} \cup \bigcup_\alpha L_\alpha$, *where the union is over all α for which* $\dim L_\alpha > d + p(\alpha)$.

Furthermore,

4. *If X is compact Kähler, then G is 1-formal, and each L_α is 1-isotropic.*
5. *If X is a smooth, quasi-projective variety, and $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Theorem (Dimca–Papadima–S. 2009)

The following are equivalent:

1. G_Γ *is a quasi-Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. G_Γ *is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Example

Let Γ be a linear path on 4 vertices. The maximal disconnected subgraphs are $\Gamma_{\{134\}}$ and $\Gamma_{\{124\}}$. Thus:

$$\mathcal{R}_1(G_\Gamma, \mathbb{C}) = \mathbb{C}^{\{134\}} \cup \mathbb{C}^{\{124\}}.$$

But $\mathbb{C}^{\{134\}} \cap \mathbb{C}^{\{124\}} = \mathbb{C}^{\{14\}}$, which is a non-zero subspace. Thus, G_Γ is not a quasi-Kähler group.

Theorem (D.–P.–S. 2008)

For a Bestvina–Brady group $N_\Gamma = \ker(\nu\colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, the following are equivalent:

1. *N_Γ is a quasi-Kähler group*
2. *Γ is either a tree, or $\Gamma = K_{n_1,\dots,n_r}$, with some $n_i = 1$, or all $n_i \geq 2$ and $r \geq 3$.*

1. *N_Γ is a Kähler group*
2. *$\Gamma = K_{2r+1}$*
3. *$N_\Gamma = \mathbb{Z}^{2r}$*

Example

$\Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2$
$N_\Gamma =$ the Stallings group $= \pi_1(\mathbb{CP}^2 \setminus \{6 \text{ lines}\})$
N_Γ is finitely presented, but $H_3(N_\Gamma, \mathbb{Z})$ has infinite rank, so N_Γ not FP_3.

References

- G. Denham, A. Suciu, *Moment-angle complexes, monomial ideals, and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, 25–60.
- A. Dimca, S. Papadima, A. Suciu, *Quasi-Kähler Bestvina-Brady groups*, J. Algebraic Geom. **17** (2008), no. 1, 185–197.
- ____, *Topology and geometry of cohomology jump loci*, Duke Math. Journal **148** (2009), no. 3, 405–457.
- S. Papadima, A. Suciu, *Algebraic invariants for right-angled Artin groups*, Math. Annalen **334** (2006), no. 3, 533–555.
- ____, *Algebraic invariants for Bestvina-Brady groups*, J. London Math. Soc. **76** (2007), no. 2, 273–292.
- ____, *Toric complexes and Artin kernels*, Adv. Math. **220** (2009), no. 2, 441–477.