Geometry and topology of cohomology jumping loci

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Characteristic varieties

- X connected CW-complex with finite k-skeleton ($k \geq 1$)
- $G = \pi_1(X, x_0)$: a finitely generated group
- $\operatorname{Hom}(G, \mathbb{C}^\times)$ character variety

Definition

For $0 \leq i \leq k$ and $d > 0$, set

$$\mathcal{V}^i_d(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^\times) \mid \dim_{\mathbb{C}} H_i(X, \mathbb{C}_\rho) \geq d\},$$

where $\mathbb{C}_\rho$ is the rank 1 local system defined by ρ, i.e, $\mathbb{C}$ viewed as a module over $\mathbb{Z}G$, via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{C}_\rho) = H_i(C_*(\widetilde{X}) \otimes_{\mathbb{Z}G} \mathbb{C}_\rho)$.

- For each i, get stratification $\operatorname{Hom}(G, \mathbb{C}^\times) \supseteq \mathcal{V}^i_1 \supseteq \mathcal{V}^i_2 \supseteq \cdots$
- Note: at $\rho = 1$, $H_i(X, \mathbb{C}_\rho) = H_i(X, \mathbb{C})$. Thus,
$$1 \in \mathcal{V}^i_1(X) \iff b_i(X) \neq 0$$
- Note: $\mathcal{V}_d(X) = \mathcal{V}^1_d(X)$ depends only on G. Write it as $\mathcal{V}_d(G)$.

Example (Circle)

We have $\widetilde{S^1} = \mathbb{R}$.
Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}[t^{\pm 1}]$. Then:

$$C_*(\widetilde{S^1}) : \; 0 \longrightarrow \mathbb{Z}[t^{\pm 1}] \xrightarrow{t-1} \mathbb{Z}[t^{\pm 1}] \longrightarrow 0$$

For $\rho \in \text{Hom}(\mathbb{Z}, C^\times) = \mathbb{C}^\times$, get

$$C_*(\widetilde{S^1}) \otimes_{\mathbb{Z}\mathbb{Z}} \mathbb{C}_\rho : \; 0 \longrightarrow \mathbb{C} \xrightarrow{\rho-1} \mathbb{C} \longrightarrow 0$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{C}) = H_1(S^1, \mathbb{C}) = \mathbb{C}$.
Hence:

$$\mathcal{V}^0_1(S^1) = \mathcal{V}^1_1(S^1) = \{1\}$$
$$\mathcal{V}^i_d(S^1) = \emptyset, \quad \text{otherwise.}$$

Example (Torus)

Identify $\pi_1(T^n) = \mathbb{Z}^n$, and $\mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$\mathcal{V}^i_d(T^n) = \begin{cases} \{1\} & \text{if } d \le \binom{n}{i}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example (Wedge of circles)

Identify $\pi_1(\bigvee^n S^1) = F_n$, and $\mathrm{Hom}(F_n, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$. Then:

$$\mathcal{V}^1_d(\bigvee^n S^1) = \begin{cases} (\mathbb{C}^\times)^n & \text{if } d < n, \\ \{1\} & \text{if } d = n, \\ \emptyset & \text{if } d > n. \end{cases}$$

Example (Orientable surface of genus $g > 1$)

$$\mathcal{V}^1_d(\Sigma_g) = \begin{cases} (\mathbb{C}^\times)^{2g} & \text{if } d < 2g-1, \\ \{1\} & \text{if } d = 2g-1, 2g, \\ \emptyset & \text{if } d > 2g. \end{cases}$$

Alexander polynomial

- $G = \pi_1(X, x_0)$
- $X^{\mathrm{ab}} \xrightarrow{p} X$ maximal torsion-free abelian cover, defined by $G \xrightarrow{\mathrm{ab}} H = H_1(G)/\mathrm{tors} \cong \mathbb{Z}^n$
- $A_G = H_1(X^{\mathrm{ab}}, p^{-1}(x_0); \mathbb{Z})$ Alex. module / $\mathbb{Z}H \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$
- $\Delta_G = \gcd(E_1(A_G))$

Proposition (Dimca–Papadima–S.)

$$\check{\mathcal{V}}_1(G) \setminus \{1\} = V(\Delta_G) \setminus \{1\},$$

where

- $\check{\mathcal{V}}_1(G) =$ *union of codim.* 1 *components of* $\mathcal{V}_1(G) \cap \mathrm{Hom}(G, \mathbb{C}^\times)^0$
- $V(\Delta_G) =$ *hypersurface in* $\mathrm{Hom}(G, \mathbb{C}^\times)^0$ *defined by* Δ_G.

Example

Let K be a non-trivial knot, $G = \pi_1(S^3 \setminus K)$. Then:

$$\mathcal{V}_1(G) = \{z \in \mathbb{C} \mid \Delta_G(z) = 0\} \cup \{1\}.$$

Tangent cones and exponential tangent cones

The homomorphism $\mathbb{C} \to \mathbb{C}^{\times}$, $z \mapsto e^z$ induces

$$\exp\colon \operatorname{Hom}(G, \mathbb{C}) \to \operatorname{Hom}(G, \mathbb{C}^{\times}), \quad \exp(0) = 1$$

Let $W = V(I)$ be a Zariski closed subset in $\operatorname{Hom}(G, \mathbb{C}^{\times})$.

Definition

- The *tangent cone* at 1 to W:

$$TC_1(W) = V(\operatorname{in}(I))$$

- The *exponential tangent cone* at 1 to W:

$$\tau_1(W) = \{z \in \operatorname{Hom}(G, \mathbb{C}) \mid \exp(tz) \in W,\ \forall t \in \mathbb{C}\}$$

Both types of tangent cones

- are homogeneous subvarieties of $\mathrm{Hom}(G, \mathbb{C})$
- are non-empty iff $1 \in W$
- depend only on the analytic germ of W at 1
- commute with finite unions and arbitrary intersections

Moreover,

- $\tau_1(W) \subseteq TC_1(W)$
  - $=$ if all irred components of W are subtori
  - $\neq$ in general
- $\tau_1(W)$ is a finite union of rationally defined subspaces

Bieri–Neumann–Strebel–Renz invariants

G finitely generated group $\rightsquigarrow \mathcal{C}(G)$ Cayley graph.
$\chi\colon G \to \mathbb{R}$ homomorphism $\rightsquigarrow \mathcal{C}_\chi(G)$ induced subgraph on vertex set $G_\chi = \{g \in G \mid \chi(g) \geq 0\}$.

Definition

$\Sigma^1(G) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \mathcal{C}_\chi(G) \text{ is connected}\}$

An open, conical subset of $\operatorname{Hom}(G, \mathbb{R}) = H^1(G, \mathbb{R})$, independent of choice of generating set for G.

Definition

$\Sigma^k(G, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid \text{the monoid } G_\chi \text{ is of type } \mathrm{FP}_k\}$

Here, G is of type FP_k if there is a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$, with P_i finitely generated for all $i \leq k$.

- The BNSR invariants $\Sigma^q(G,\mathbb{Z})$ form a descending chain of *open* subsets of $\text{Hom}(G,\mathbb{R}) \setminus \{0\}$.
- $\Sigma^k(G,\mathbb{Z}) \neq \emptyset \implies G$ is of type FP_k.
- $\Sigma^1(G,\mathbb{Z}) = \Sigma^1(G)$.
- The Σ-invariants control the finiteness properties of normal subgroups $N \triangleleft G$ with G/N is abelian:

$$N \text{ is of type } \text{FP}_k \Longleftrightarrow S(G,N) \subseteq \Sigma^k(G,\mathbb{Z})$$

  where $S(G,N) = \{\chi \in \text{Hom}(G,\mathbb{R}) \setminus \{0\} \mid \chi(N) = 0\}$.
- In particular:

$$\ker(\chi\colon G \twoheadrightarrow \mathbb{Z}) \text{ is f.g.} \Longleftrightarrow \{\pm\chi\} \subseteq \Sigma^1(G)$$

Let X be a connected CW-complex with finite 1-skeleton, $G = \pi_1(X)$.

Definition

The *Novikov-Sikorav completion* of $\mathbb{Z}G$:

$$\widehat{\mathbb{Z}G}_\chi = \left\{ \lambda \in \mathbb{Z}^G \mid \{g \in \operatorname{supp}\lambda \mid \chi(g) < c\} \text{ is finite, } \forall c \in \mathbb{R} \right\}$$

$\widehat{\mathbb{Z}G}_\chi$ is a ring, contains $\mathbb{Z}G$ as a subring $\implies$ $\widehat{\mathbb{Z}G}_\chi$ is a $\mathbb{Z}G$-module.

Definition

$$\Sigma^q(X, \mathbb{Z}) = \{\chi \in \operatorname{Hom}(G, \mathbb{R}) \setminus \{0\} \mid H_i(X, \widehat{\mathbb{Z}G}_{-\chi}) = 0,\ \forall i \le q\}$$

Bieri: G of type FP_k $\implies$ $\Sigma^q(G, \mathbb{Z}) = \Sigma^q(K(G,1), \mathbb{Z})$, $\forall q \le k$.

Exponential tangent cone upper bound

Theorem (Papadima–S.)

If X has finite k-skeleton, then, for every $q \leq k$,

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\tau_1^{\mathbb{R}}\big(\bigcup_{i\leq q} \mathcal{V}_1^i(X)\big)\Big)^{\complement}. \qquad (*)$$

Thus: Each Σ-invariant is contained in the complement of a union of rationally defined subspaces. Bound is sharp:

Example

Let G be a finitely generated nilpotent group. Then

$$\Sigma^q(G,\mathbb{Z}) = \operatorname{Hom}(G,\mathbb{R}) \setminus \{0\}, \quad V_1^q(G) = \{1\}, \quad \forall q$$

and so (*) holds as an equality.

Resonance varieties

Let X be a connected CW-complex with finite k-skeleton ($k \geq 1$).
Let $A = H^*(X, \mathbb{C})$. Then: $a \in A^1 \Rightarrow a^2 = 0$. Thus, get cochain complex

$$(A, \cdot a)\colon \; A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \longrightarrow \cdots$$

Definition

The *resonance varieties* of X are the algebraic sets

$$\mathcal{R}^i_d(X) = \{a \in A^1 \mid \dim_{\Bbbk} H^i(A, a) \geq d\},$$

defined for all integers $0 \leq i \leq k$ and $d > 0$.

- $\mathcal{R}^i_d$ are homogeneous subvarieties of $A^1 = H^1(X, \mathbb{C})$
- $\mathcal{R}^i_1 \supseteq \mathcal{R}^i_2 \supseteq \cdots \supseteq \mathcal{R}^i_{b_i+1} = \emptyset$, where $b_i = b_i(X)$.
- $\mathcal{R}_d(X) = \mathcal{R}^1_d(X)$ depends only on $G = \pi_1(X)$. Write as $\mathcal{R}_d(G)$.

Equivalent definition:

$$\mathcal{R}_d(X) = \left\{ a \in H^1(X,\mathbb{C}) \;\middle|\; \begin{array}{l} \exists \text{ subspace } W \subset H^1(X,\mathbb{C}) \text{ such that} \\ \dim W = d+1 \text{ and } a \cup W = 0 \end{array} \right\}$$

In particular, $0 \neq a \in H^1(X,\mathbb{C})$ belongs to $\mathcal{R}_1(X)$ iff there is $b \in H^1(X)$ not proportional to a, such that $a \cup b = 0$ in $H^2(X)$.

Example

- $\mathcal{R}_1(T^n) = \{0\}$, for all $n > 0$.
- $\mathcal{R}_1(\bigvee^n S^1) = \mathbb{C}^n$, for all $n > 1$.
- $\mathcal{R}_1(\Sigma_g) = \mathbb{C}^{2g}$, for all $g > 1$.

Theorem (Libgober 2002)

$$TC_1(\mathcal{V}^i_d(X)) \subseteq \mathcal{R}^i_d(X)$$

Equality does not hold in general (Matei–S. 2002)

Formality

Definition

1. A group G is 1-*formal* if its Malcev Lie algebra, $\mathfrak{m}_G = \mathrm{Prim}(\widehat{\mathbb{Q}G})$, is quadratic.
2. A space X is *formal* if its minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$.

- X formal $\implies$ $\pi_1(X)$ is 1-formal.
- X_1, X_2 formal $\implies$ $X_1 \times X_2$ and $X_1 \vee X_2$ are formal
- G_1, G_2 1-formal $\implies$ $G_1 \times G_2$ and $G_1 * G_2$ are 1-formal
- M_1, M_2 formal, closed n-manifolds $\implies$ $M_1 \# M_2$ formal

Tangent cone theorem

Theorem (Dimca–Papadima–S.)

If G is 1-formal, then $\exp\colon (\mathcal{R}_d(G),0) \xrightarrow{\simeq} (\mathcal{V}_d(G,\mathbb{C}),1)$. *Hence*

$$\tau_1(\mathcal{V}_d(G)) = TC_1(\mathcal{V}_d(G)) = \mathcal{R}_d(G)$$

In particular, $\mathcal{R}_d(G)$ is a union of rationally defined linear subspaces in $H^1(G,\mathbb{C}) = \mathrm{Hom}(G,\mathbb{C})$.

Example

Let $G = \langle x_1, x_2, x_3, x_4 \mid [x_1,x_2], [x_1,x_4][x_2^{-2},x_3], [x_1^{-1},x_3][x_2,x_4]\rangle$. Then

$$\mathcal{R}_1(G) = \{x \in \mathbb{C}^4 \mid x_1^2 - 2x_2^2 = 0\}$$

splits into subspaces over $\mathbb{R}$ but not over $\mathbb{Q}$. Thus, G is *not* 1-formal.

Example

- $X = F(\Sigma_g, n)$: the configuration space of n labeled points of a Riemann surface of genus g (a smooth, quasi-projective variety).
- $\pi_1(X) = P_{g,n}$: the pure braid group on n strings on Σ_g.

Using computation of $H^*(F(\Sigma_g, n), \mathbb{C})$ by Totaro (1996), get

$$\mathcal{R}_1(P_{1,n}) = \left\{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \;\middle|\; \begin{array}{l} \sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0, \\ x_i y_j - x_j y_i = 0, \text{ for } 1 \le i < j < n \end{array} \right\}$$

For $n \geq 3$, this is an irreducible, non-linear variety (a rational normal scroll). Hence, $P_{1,n}$ is not 1-formal.

Resonance upper bound

Corollary

Suppose $\exp\colon (\mathcal{R}^i_1(X), 0) \xrightarrow{\simeq} (\mathcal{V}^i_1(X), 1)$, *for* $i \le q$. *Then:*

$$\Sigma^q(X,\mathbb{Z}) \subseteq \Big(\bigcup_{i\le q} \mathcal{R}^i_1(X,\mathbb{R}) \Big)^{\mathsf{c}}.$$

Corollary

Suppose G *is a* 1*-formal group. Then* $\Sigma^1(G) \subseteq \mathcal{R}_1(G,\mathbb{R})^{\mathsf{c}}$*.*
In particular, if $\mathcal{R}_1(G,\mathbb{R}) = H^1(G,\mathbb{R})$*, then* $\Sigma^1(G) = \emptyset$*.*

Example

The above inclusion may be strict: Let $G = \langle a, b \mid aba^{-1} = b^2 \rangle$.
Then G is 1-formal, $\Sigma^1(G) = (-\infty, 0)$, yet $\mathcal{R}_1(G,\mathbb{R}) = \{0\}$.

Kähler manifolds and Kähler groups

Definition

A compact, connected, complex manifold M is called a *Kähler manifold* if M admits a Hermitian metric h for which the imaginary part $\omega = \Im(h)$ is a closed 2-form.

Examples: Riemann surfaces, $\mathbb{CP}^n$, and, more generally, smooth, complex projective varieties.

Definition

A group G is a *Kähler group* if $G = \pi_1(M)$, for some compact Kähler manifold M.

G is *projective* if M is actually a smooth projective variety.

- G finite $\Rightarrow$ G is a projective group (Serre 1958).
- G_1, G_2 Kähler groups $\Rightarrow$ $G_1 \times G_2$ is a Kähler group
- G Kähler group, $H < G$ finite-index subgroup $\Rightarrow$ H is a Kähler gp

Problem (J.-P. Serre 1958)

Which finitely presented groups are Kähler (or projective) groups?

The Kähler condition puts strong restrictions on M:

1. $H^*(M, \mathbb{Z})$ admits a Hodge structure
2. Hence, the odd Betti numbers of M are even
3. M is formal (Deligne–Griffiths–Morgan–Sullivan 1975)

The Kähler condition also puts strong restrictions on $G = \pi_1(M)$:

1. $b_1(G)$ is even
2. G is 1-formal
3. G cannot split non-trivially as a free product (Gromov 1989)

Quasi-Kähler manifolds

Definition

A manifold X is called *quasi-Kähler* if $X = \overline{X} \setminus D$, where $\overline{X}$ is Kähler and D is a divisor with normal crossings.

- Smooth quasi-projective varieties (e.g., complements of hypersurfaces in $\mathbb{CP}^n$) are quasi-Kähler manifolds.
- A finitely-presented group G is called a *quasi-Kähler group* if there a quasi-Kähler manifold X with $G = \pi_1(X)$.
- $X = \mathbb{CP}^n \setminus \{\text{hyperplane arrangement}\} \Rightarrow X$ is formal
  (Brieskorn 1973)
- X quasi-projective, $W_1(H^1(X, \mathbb{C})) = 0 \Rightarrow \pi_1(X)$ is 1-formal
  (Morgan 1978)
- $X = \mathbb{CP}^n \setminus \{\text{hypersurface}\} \Rightarrow \pi_1(X)$ is 1-formal
  (Kohno 1983)

Theorem (Arapura 1997)

Let G be a quasi-Kähler group. Then

$$\mathcal{V}_1(G) = \bigcup_{\alpha} \rho_\alpha T_\alpha$$

where T_α is an algebraic subtorus of $\operatorname{Hom}(G, \mathbb{C}^\times)$ *and ρ_α is a finite-order character.*

Theorem (Dimca–Papadima–S.)

Let G be a quasi-Kähler group, and Δ_G its Alexander polynomial.

- *If $b_1(G) \neq 2$, then the Newton polytope of Δ_G is a line segment.*
- *If G is actually a Kähler group, then $\Delta_G \doteq \text{const}$.*

Resonance varieties of quasi-Kähler manifolds

Theorem (Dimca–Papadima–S.)

Let X be a quasi-Kähler manifold, and $G = \pi_1(X)$. Let $\{L_\alpha\}_\alpha$ be the non-zero irred components of $\mathcal{R}_1(G)$. If G is 1-formal, then

1. *Each L_α is a linear subspace of $H^1(G, \mathbb{C})$.*
2. *Each L_α is p-isotropic (i.e., restriction of $\cup_G$ to L_α has rank p), with $\dim L_\alpha \geq 2p + 2$, for some $p = p(\alpha) \in \{0, 1\}$.*
3. *If $\alpha \neq \beta$, then $L_\alpha \cap L_\beta = \{0\}$.*
4. $\mathcal{R}_d(G) = \{0\} \cup \bigcup_{\alpha : \dim L_\alpha > d + p(\alpha)} L_\alpha$.

Furthermore,

4. *If X is compact, then G is 1-formal, and each L_α is 1-isotropic.*
5. *If $W_1(H^1(X, \mathbb{C})) = 0$, then G is 1-formal, and each L_α is 0-isotropic.*

Σ-invariants

Let X be a quasi-Kähler manifold, $G = \pi_1(X)$.

Theorem (Papadima–S.)

1. $\Sigma^1(G) \subseteq TC_1^{\mathbb{R}}(\mathcal{V}_1^1(G))^{\complement}$.
2. *If X is Kähler, or $W_1(H^1(X, \mathbb{C})) = 0$, then $\mathcal{R}_1(G, \mathbb{R})$ is a finite union of rationally defined linear subspaces, and $\Sigma^1(G) \subseteq \mathcal{R}_1(G, \mathbb{R})^{\complement}$.*

Example

Assumption from (2) is necessary. E.g., let X be the complex Heisenberg manifold: bundle $\mathbb{C}^\times \to X \to (\mathbb{C}^\times)^2$ with $e = 1$. Then:

1. X is a smooth quasi-projective variety;
2. $G = \pi_1(X)$ is nilpotent (and not 1-formal);
3. $\Sigma^1(G) = \mathbb{R}^2 \setminus \{0\}$ and $\mathcal{R}_1(G, \mathbb{R}) = \mathbb{R}^2$.

Thus, $\Sigma^1(G) \not\subseteq \mathcal{R}_1(G, \mathbb{R})^{\complement}$.

Toric complexes and right-angled Artin groups

- L simplicial complex on n vertices $\rightsquigarrow$ *toric complex* T_L (subcomplex of T^n obtained by deleting the cells corresponding to the missing simplices of L).
- $\pi_1(T_L)$ is the *right-angled Artin group* associated to graph $\Gamma = L^{(1)}$:

$$G_\Gamma = \langle v \in V(\Gamma) \mid vw = wv \text{ if } \{v, w\} \in E(\Gamma)\rangle.$$

- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the *flag complex* of Γ.
- $H^*(T_L, \mathbb{Z})$ is the *exterior Stanley-Reisner ring* of L, with generators the duals v^*, and relations the monomials corresponding to the missing simplices of L.
- T_L is formal, and so G_Γ is 1-formal.

Example

- $\Gamma = \overline{K}_n \Rightarrow G_\Gamma = F_n$
- $\Gamma = K_n \Rightarrow G_\Gamma = \mathbb{Z}^n$
- $\Gamma = \Gamma' \coprod \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} * G_{\Gamma''}$
- $\Gamma = \Gamma' * \Gamma'' \Rightarrow G_\Gamma = G_{\Gamma'} \times G_{\Gamma''}$

Identify $H^1(T_L,\mathbb{C}) = \mathbb{C}^{\mathsf{V}}$, the $\mathbb{C}$-vector space with basis $\{v \mid v \in \mathsf{V}\}$.

Theorem (Papadima–S.)

$$\mathcal{R}^i_d(T_L) = \bigcup_{\substack{\mathsf{W}\subset\mathsf{V} \\ \sum_{\sigma\in L_{\mathsf{V}\setminus\mathsf{W}}} \dim_{\mathbb{C}} \widetilde{H}_{i-1-|\sigma|}(\operatorname{lk}_{L_{\mathsf{W}}}(\sigma),\mathbb{C})\geq d}} \mathbb{C}^{\mathsf{W}},$$

where L_{W} is the subcomplex induced by L on W, and $\operatorname{lk}_K(\sigma)$ is the link of a simplex σ in a subcomplex $K \subseteq L$.

In particular:

$$\mathcal{R}_1(G_\Gamma) = \bigcup_{\substack{\mathsf{W}\subseteq\mathsf{V} \\ \Gamma_{\mathsf{W}} \text{ disconnected}}} \mathbb{C}^{\mathsf{W}}.$$

Similar formula holds for $\mathcal{V}^i_d(T_L)$, with $\mathbb{C}^{\mathsf{W}}$ replaced by $(\mathbb{C}^\times)^{\mathsf{W}}$. Hence:

$$\exp\colon (\mathcal{R}^i_d(T_L), 0) \xrightarrow{\simeq} (\mathcal{V}^i_d(T_L), 1).$$

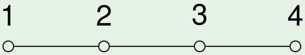
Using (1) resonance upper bound, and (2) computation of $\Sigma^k(G_\Gamma, \mathbb{Z})$ by Meier, Meinert, VanWyk (1998), we get:

Corollary (Papadima-S.)

$$\Sigma^k(T_L, \mathbb{Z}) \subseteq \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_L, \mathbb{R})\Big)^{\mathsf{C}}$$

$$\Sigma^k(G_\Gamma, \mathbb{Z}) = \Big(\bigcup_{i\le k} \mathcal{R}^i_1(T_{\Delta_\Gamma}, \mathbb{R})\Big)^{\mathsf{C}}$$

Example

$\Gamma =$ 

Maximal disconnected subgraphs: $\Gamma_{\{134\}}$ and $\Gamma_{\{124\}}$. Thus:

$$\mathcal{R}_1(G_\Gamma) = \mathbb{C}^{\{134\}} \cup \mathbb{C}^{\{124\}}.$$

Note that: $\mathbb{C}^{\{134\}} \cap \mathbb{C}^{\{124\}} = \mathbb{C}^{\{14\}} \neq \{0\}$
Since G_Γ is 1-formal $\Rightarrow$ G_Γ is not a quasi-Kähler group.

Theorem (Dimca–Papadima–S.)

The following are equivalent:

1. *G_Γ is a quasi-Kähler group*
2. $\Gamma = K_{n_1,\dots,n_r} := \overline{K}_{n_1} * \cdots * \overline{K}_{n_r}$
3. $G_\Gamma = F_{n_1} \times \cdots \times F_{n_r}$

1. *G_Γ is a Kähler group*
2. $\Gamma = K_{2r}$
3. $G_\Gamma = \mathbb{Z}^{2r}$

Bestvina–Brady groups

$N_\Gamma = \ker(\nu \colon G_\Gamma \twoheadrightarrow \mathbb{Z})$, where $\nu(v) = 1$, for all $v \in V(\Gamma)$.

Theorem (Dimca–Papadima–S.)

The following are equivalent:

1. N_Γ *is a quasi-Kähler group*
2. Γ *is either a tree, or* $\Gamma = K_{n_1,\dots,n_r}$, *with some* $n_i = 1$, *or all* $n_i \geq 2$ *and* $r \geq 3$.

1. N_Γ *is a Kähler group*
2. $\Gamma = K_{2r+1}$
3. $N_\Gamma = \mathbb{Z}^{2r}$

Example (answers a question of J. Kollár)

$\Gamma = K_{2,2,2} \rightsquigarrow G_\Gamma = F_2 \times F_2 \times F_2 \rightsquigarrow N_\Gamma =$ the Stallings group

N_Γ is finitely presented, but rank $H_3(N_\Gamma, \mathbb{Z}) = \infty$, so N_Γ not FP_3.

Also, $N_\Gamma = \pi_1(\mathbb{C}^2 \setminus \{\text{an arrangement of 5 lines}\})$.

Thus, N_Γ is a quasi-projective group which is not commensurable (even up to finite kernels) to any group π having a finite $K(\pi, 1)$.

3-manifolds

Question (Donaldson–Goldman 1989, Reznikov 1993)

Which 3-manifold groups are Kähler groups?

Reznikov (2002) and Hernández-Lamoneda (2001) gave partial solutions.

Theorem (Dimca–S.)

Let G be the fundamental group of a closed 3-manifold. Then G is a Kähler group $\iff$ G is a finite subgroup of $\mathrm{O}(4)$, acting freely on S^3.

Idea of proof: compare the resonance varieties of (orientable) 3-manifolds to those of Kähler manifolds.

Proposition

Let M be a closed, orientable 3-manifold. Then:

1. $H^1(M,\mathbb{C})$ *is not* 1*-isotropic.*
2. *If* $b_1(M)$ *is even, then* $\mathcal{R}_1(M) = H^1(M,\mathbb{C})$.

On the other hand, it follows from [Dimca–Papadima–S.] that:

Proposition

Let M be a compact Kähler manifold with $b_1(M) \neq 0$. *If* $\mathcal{R}_1(M) = H^1(M,\mathbb{C})$, *then* $H^1(M,\mathbb{C})$ *is* 1*-isotropic.*

But $G = \pi_1(M)$, with M Kähler $\Rightarrow b_1(G)$ even.
Thus, if G is both a 3-mfd group and a Kähler group $\Rightarrow b_1(G) = 0$.
Using work of Fujiwara (1999) and Reznikov (2002) on Kazhdan's property (T), as well as Perelman (2003) $\Rightarrow G$ finite subgroup of O(4).

Question

Which 3-manifolds groups are quasi-Kähler groups?

Theorem (Dimca–Papadima–S.)

Let G be the fundamental group of a closed, orientable 3-manifold. Assume G is 1-formal. Then the following are equivalent:

1. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(X))$*, for some quasi-Kähler manifold* X*.*
2. $\mathfrak{m}(G) \cong \mathfrak{m}(\pi_1(M))$*, where* M *is either* S^3*,* $\#^n S^1 \times S^2$*, or* $S^1 \times \Sigma_g$*.*

BNS invariant and Thurston norm

Let M be a compact, connected 3-manifold, with $G = \pi_1(M)$.

Theorem (Bieri–Neumann–Strebel 1987)

- $\Sigma^1(G) = \bigcup_{F \text{ fibered face of Thurston norm unit ball}} \mathbb{R}_+ \cdot \mathring{F}$.
- $\Sigma^1(G) = -\Sigma^1(G)$.
- *M fibers over S^1* $\iff \Sigma^1(G) \neq \emptyset$.

Using (1) upper bound $\Sigma^1(G) \subseteq \mathcal{R}_1(G, \mathbb{R})^{\complement}$ for 1-formal groups, and (2) description of $\mathcal{R}_1(M^3)$ from above, we get:

Corollary (Papadima–S.)

Let M be a closed, orientable 3-manifold. If $b_1(M)$ is even, and $G = \pi_1(M)$ is 1-formal, then M does not fiber over the circle.

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