MULTINETS, PARALLEL CONNECTIONS, AND MILNOR FIBRATIONS OF ARRANGEMENTS

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THE MILNOR FIBRATION OF AN ARRANGEMENT

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$, with complement $M(\mathcal{A})$.

- Let $Q_m = \prod_{H \in \mathcal{A}} f_H^{m_H}$, where $H = \ker(f_H)$ and $m \in \mathbb{N}^{|\mathcal{A}|}$: a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.

- The map $Q_m \colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a map $Q_m \colon M(\mathcal{A}) \to \mathbb{C}^*$.

- This is the projection of a smooth, locally trivial bundle, known as the *(global) Milnor fibration* of the multi-arrangement $(\mathcal{A}, m)$,

$$F_m(\mathcal{A}) \longrightarrow M(\mathcal{A}) \xrightarrow{\ Q_m\ } \mathbb{C}^*.$$

- The typical fiber, $F_m(\mathcal{A}) = Q_m^{-1}(1)$, is called the *Milnor fiber* of the multi-arrangement.

- The *(geometric) monodromy* is the diffeomorphism

$$h \colon F_m(\mathcal{A}) \to F_m(\mathcal{A}), \quad z \mapsto e^{2\pi i / N} z.$$

- $F_m(\mathcal{A})$ is a Stein domain of complex dimension d; thus, it has the homotopy type of a finite CW-complex of dimension d.
- If all $m_H = 1$, the polynomial $Q = Q_m$ is the usual defining polynomial, and $F(\mathcal{A}) = F_m(\mathcal{A})$ is the usual Milnor fiber of $\mathcal{A}$.

EXAMPLE

Let $\mathcal{A}$ be a pencil of 3 lines through the origin of $\mathbb{C}^2$. Then $F(\mathcal{A})$ is a thrice-punctured torus, and h is an automorphism of order 3:
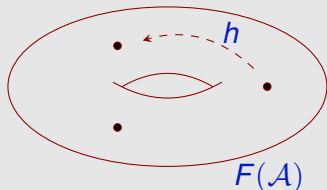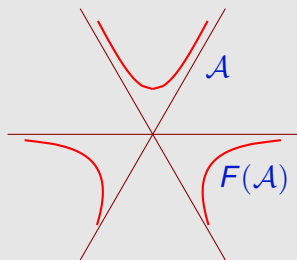
- As conjectured by Papadima–S. and proved by Dimca–Papadima, Randell, Salvetti–Settepanella, etc, the complement $M(\mathcal{A})$ admits a minimal cell structure (Morse inequalities satisfied on the nose).

- In particular, $H_*(M(\mathcal{A}), \mathbb{Z})$ is torsion-free.

- Question (Dimca–Némethi, Randell): Is the same true for $F(\mathcal{A})$?

- In previous work (Cohen–Denham–S. 2003), we showed that the answer is no, but only for $F_m(\mathcal{A})$, for non-trivial m.

- Here, we settle the question for the standard Milnor fiber $F(\mathcal{A})$.

THEOREM

For every prime $p \geqslant 2$, there is a hyperplane arrangement whose Milnor fiber has non-trivial p-torsion in homology.

In particular, Milnor fibers of arrangements do not necessarily have a minimal cell structure.
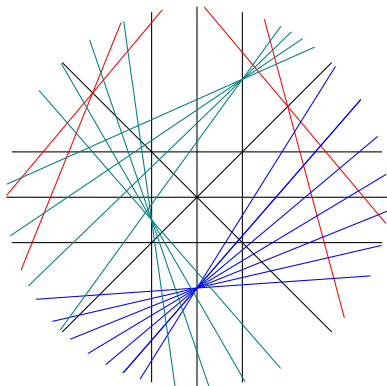
Simplest example: the arrangement of 27 hyperplanes in $\mathbb{C}^8$ with

$$Q(\mathcal{A}) = xy(x^2 - y^2)(x^2 - z^2)(y^2 - z^2)w_1 w_2 w_3 w_4 w_5(x^2 - w_1^2)\cdot$$
$$(x^2 - 2w_1^2)(x^2 - 3w_1^2)(x - 4w_1)((x - y)^2 - w_2^2)((x + y)^2 - w_3^2)\cdot$$
$$((x - z)^2 - w_4^2)((x - z)^2 - 2w_4^2)\cdot((x + z)^2 - w_5^2)((x + z)^2 - 2w_5^2).$$

Then $H_6(F(\mathcal{A}), \mathbb{Z})$ has 2-torsion (of rank 108).

Basic idea of proof:

- $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U = \mathbb{P}M(\mathcal{A})$ defined by the homomorphism $\delta_m \colon \pi_1(U) \twoheadrightarrow \mathbb{Z}_N$, taking each meridian x_H to m_H.

- If $\Bbbk = \overline{\Bbbk}$ and $\mathrm{char}(\Bbbk) \nmid N$, then $H_q(F_m(\mathcal{A}), \Bbbk) \cong \bigoplus_\rho H_q(U, \Bbbk_\rho)$, where $\rho \colon \pi_1(U) \to \Bbbk^*$ factors through δ_m.

- Thus, if there is such a character ρ for which $H_q(U, \Bbbk_\rho) \neq 0$, where $p = \mathrm{char}(\Bbbk) \nmid N$, but there is no corresponding character in characteristic 0, then $H_q(F_m(\mathcal{A}), \mathbb{Z})$ will have p-torsion.

- Fix a prime p. Starting with an arrangement $\mathcal{A}$ with a suitable multinet structure, we find a deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$, and a choice of multiplicities m' on $\mathcal{A}'$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has p-torsion.

- Finally, we construct a "polarized" arrangement, $\mathcal{B} = \mathcal{A}' \parallel m'$, and show that $H_*(F(\mathcal{B}), \mathbb{Z})$ has p-torsion.

CHARACTERISTIC VARIETIES

- Let X be a connected, finite-type CW-complex. Then $G = \pi_1(X, x_0)$ is a finitely generated group, with $G_{\mathrm{ab}} \cong H_1(X, \mathbb{Z})$.

- The character group $\widehat{G} = \widehat{G}_{\mathbb{k}} := \mathrm{Hom}(G, \mathbb{k}^*) \cong H^1(X, \mathbb{k}^*)$ is an abelian algebraic group, with $\widehat{G} \cong \widehat{G_{\mathrm{ab}}}$.

- Characteristic varieties:

$$\mathcal{V}^i_s(X, \mathbb{k}) := \{\rho \in \widehat{G} \mid \dim_{\mathbb{k}} H_i(X, \mathbb{k}_\rho) \geqslant s\}.$$

  Here, $\mathbb{k}_\rho$ is the local system defined by ρ, i.e, $\mathbb{k}$ viewed as a $\mathbb{k}G$-module, via $g \cdot x = \rho(g)x$, and $H_i(X, \mathbb{k}_\rho) = H_i(C_*(\widetilde{X}, \mathbb{k}) \otimes_{\mathbb{k}G} \mathbb{k}_\rho)$.

- Product formula: $\mathcal{V}^i_1(X_1 \times X_2, \mathbb{k}) = \bigcup_{p+q=i} \mathcal{V}^p_1(X_1, \mathbb{k}) \times \mathcal{V}^q_1(X_2, \mathbb{k})$.

- The sets $\mathcal{V}^1_s(X, \mathbb{k})$ depend only on $G = \pi_1(X)$—in fact, only on G/G''. Write them as $\mathcal{V}^1_s(G, \mathbb{k})$, and set $\mathcal{V}^1(G, \mathbb{k}) = \mathcal{V}^1_s(G, \mathbb{k})$.

- If $\varphi \colon G \twoheadrightarrow Q$ is an epimorphism, then the induced morphism $\hat{\varphi}_{\mathbb{k}} \colon \widehat{Q} \hookrightarrow \widehat{G}$ restricts to an embedding $\mathcal{V}^1_s(Q, \mathbb{k}) \hookrightarrow \mathcal{V}^1_s(G, \mathbb{k})$, $\forall s$.

EXAMPLE

We have $\widetilde{S^1} = \mathbb{R}$. Identify $\pi_1(S^1, *) = \mathbb{Z} = \langle t \rangle$ and $\mathbb{k}\mathbb{Z} = \mathbb{k}[t^{\pm 1}]$.
Then:
$$C_*(\widetilde{S^1}, \mathbb{k}) : \ 0 \longrightarrow \mathbb{k}[t^{\pm 1}] \xrightarrow{\ t-1\ } \mathbb{k}[t^{\pm 1}] \longrightarrow 0 \ .$$

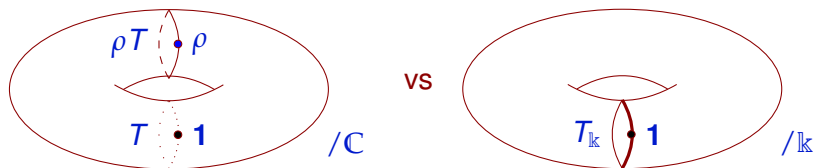For $\rho \in \mathsf{Hom}(\mathbb{Z}, \mathbb{k}^*) = \mathbb{k}^*$, we get

$$C_*(\widetilde{S^1}, \mathbb{k}) \otimes_{\mathbb{k}\mathbb{Z}} \mathbb{k}_\rho : \ 0 \longrightarrow \mathbb{k} \xrightarrow{\ \rho-1\ } \mathbb{k} \longrightarrow 0 \ ,$$

which is exact, except for $\rho = 1$, when $H_0(S^1, \mathbb{k}) = H_1(S^1, \mathbb{k}) = \mathbb{k}$.
Hence: $\mathcal{V}_1^0(S^1, \mathbb{k}) = \mathcal{V}_1^1(S^1, \mathbb{k}) = \{1\}$ and $\mathcal{V}_s^i(S^1, \mathbb{k}) = \varnothing$, otherwise.

EXAMPLE

Identify $\pi_1(\mathbb{C}\backslash\{n \text{ points}\}) = F_n$, and $\widehat{F_n} = (\mathbb{k}^*)^n$. Then:
$$\mathcal{V}_s^1(\mathbb{C}\backslash\{n \text{ points}\}, \mathbb{k}) = \begin{cases} (\mathbb{k}^*)^n & \text{if } s < n, \\ \{1\} & \text{if } s = n, \\ \varnothing & \text{if } s > n. \end{cases}$$

In general, though, the characteristic varieties of a space (or a group) will depend on the characteristic of the ground field.

EXAMPLE

- Let $\Gamma = F_n * \mathbb{Z}_p$, and identify $\widehat{\Gamma} = \widehat{F_n} \times \widehat{\mathbb{Z}_p}$.

- If $\mathrm{char}(\Bbbk) \neq p$, then $\widehat{\mathbb{Z}_p} = \mathrm{Hom}(\mathbb{Z}_p, \Bbbk^*)$ has order p, and

$$\mathcal{V}^1(\Gamma, \Bbbk) = \begin{cases} \widehat{\Gamma} & \text{if } n \geqslant 2, \\ (\widehat{\Gamma} \backslash \widehat{\Gamma}^\circ) \cup \{\mathbf{1}\} & \text{if } n = 1 . \end{cases}$$

- If $\mathrm{char}(\Bbbk) = p$, then $\widehat{\mathbb{Z}_p} = \{\mathbf{1}\}$, and $\mathcal{V}^1(\Gamma, \Bbbk) = \widehat{\Gamma}$, for all $n \geqslant 1$.

HOMOLOGY OF FINITE ABELIAN COVERS

- Let X be a connected, finite-type CW-complex, and $G = \pi_1(X)$.

- Let A be a finite abelian group.

- Every epimorphism $\chi \colon G \to A$ determines a regular, connected A-cover $X^\chi \to X$.

- Let $\Bbbk$ be a field, $p = \operatorname{char}(\Bbbk)$. Assume $p = 0$ or $p \nmid |A|$. Then
$$H_q(X^\chi, \Bbbk) \cong H_q(X, \Bbbk[A]) \cong \bigoplus_{\rho \in \widehat{A}} H_q(X, \Bbbk_\rho).$$

Hence,

THEOREM (LIBGOBER, SAKUMA, HIRONAKA)

$$\dim_\Bbbk H_q(X^\chi, \Bbbk) = \sum_{s \geqslant 1} \left| \mathcal{V}^q_s(X, \Bbbk) \cap \operatorname{im}(\widehat{\chi}_\Bbbk) \right|,$$

where $\widehat{\chi}_\Bbbk \colon \widehat{A}_\Bbbk \to \widehat{G}_\Bbbk$ is the induced morphism between character groups.

- Now suppose A is a finite cyclic group. Choose a generator α of A.

- Let $h = h_\alpha \colon X^\chi \to X^\chi$ be the monodromy automorphism.

- Let $h_* \colon H_q(X^\chi, \Bbbk) \to H_q(X^\chi, \Bbbk)$ be the induced homomorphism.

- Note: if $\rho \colon G \to \Bbbk^*$ is a character belonging to $\operatorname{im}(\widehat{\chi}_\Bbbk)$, there is a unique character $\iota_\rho \colon A \to \Bbbk^*$ such that $\rho = \iota_\rho \circ \chi$.

THEOREM

Assume $\operatorname{char}(\Bbbk) \nmid |A|$. *Then, the characteristic polynomial of the algebraic monodromy,* $\Delta^{\Bbbk}_{\chi,q}(t) = \det(t \cdot \operatorname{id} - h_*)$, *is given by*

$$\Delta^{\Bbbk}_{\chi,q}(t) = \prod_{s \geqslant 1} \prod_{\rho \in \operatorname{im}(\widehat{\chi}) \cap \mathcal{V}^q_s(X, \Bbbk)} (t - \iota_\rho(\alpha)).$$

THEOREM

Let $X^\chi \to X$ be a regular, finite cyclic cover, defined by an epimorphism $\chi \colon \pi_1(X) \twoheadrightarrow \mathbb{Z}_r$. Suppose that $\operatorname{im}(\widehat{\chi}_{\mathbb{C}}) \nsubseteq \mathcal{V}_1^q(X, \mathbb{C})$, but $\operatorname{im}(\widehat{\chi}_{\mathbb{k}}) \subseteq \mathcal{V}_1^q(X, \mathbb{k})$, for some field $\mathbb{k}$ of characteristic p not dividing r. Then $H_q(X^\chi, \mathbb{Z})$ has non-zero p-torsion.

- In order to apply this theorem, one needs to know both the characteristic varieties over $\mathbb{C}$ and over $\mathbb{k}$, and exploit the qualitative differences between the two.
- In a special situation, we only need to verify that a certain condition on the first characteristic variety over $\mathbb{C}$ holds.

THEOREM

Suppose there is a character $\rho \colon \pi_1(X) \to \mathbb{C}^*$ which factors as $\pi_1(X) \twoheadrightarrow \mathbb{Z} * \mathbb{Z}_p \twoheadrightarrow \mathbb{Z} \to \mathbb{C}^*$, for some prime p, but $\rho \notin \mathcal{V}^1(X, \mathbb{C})$. Then, for all sufficiently large integers r not divisible by p, there is a regular, r-fold cyclic cover $Y \to X$ such that $H_1(Y, \mathbb{Z})$ has non-zero p-torsion.

ORBIFOLD FIBRATIONS

- Let $\Sigma_{g,r}$ be a Riemann surface of genus $g \geqslant 0$ with $r \geqslant 0$ points removed. Fix points $q_1, \ldots, q_s$ on the surface, and assign integer weights $\mu_1, \ldots, \mu_s$ with $\mu_i \geqslant 2$.

- The orbifold $\Sigma = (\Sigma_{g,r}, \mu)$ is *hyperbolic* if $\chi^{\text{orb}}(\Sigma) := 2 - 2g - r - \sum_{i=1}^{s}(1 - 1/\mu_i)$ is negative.

- A hyperbolic orbifold Σ is *small* if either $\Sigma = S^1 \times S^1$ and $s \geqslant 2$, or $\Sigma = \mathbb{C}^*$ and $s \geqslant 1$; otherwise, Σ is *large*.

- Let $\Gamma := \pi_1^{\text{orb}}(\Sigma_{g,r}, \mu)$, defined as

$$\Gamma = \left\langle \begin{array}{c} x_1, \ldots, x_g, y_1, \ldots, y_g \\ z_1, \ldots, z_s \end{array} \;\middle|\; \begin{array}{c} [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_s = 1 \\ z_1^{\mu_1} = \cdots = z_t^{\mu_s} = 1 \end{array} \right\rangle (r = 0)$$

$$\Gamma = F_{2g+r-1} * \mathbb{Z}_{\mu_1} * \cdots * \mathbb{Z}_{\mu_s} \qquad (r > 0)$$

Then:

$$\mathcal{V}^1(\Gamma) = \begin{cases} \widehat{\Gamma} & \text{if } \Sigma \text{ is a large hyperbolic orbifold,} \\ (\widehat{\Gamma} \backslash \widehat{\Gamma}^\circ) \cup \{1\} & \text{if } \Sigma \text{ is a small hyperbolic orbifold,} \\ \{1\} & \text{otherwise.} \end{cases}$$

- Let X be a smooth, quasi-projective variety, and $G = \pi_1(X)$.
- A surjective, holomorphic map $f\colon X \to (\Sigma_{g,r}, \mu)$ is called an *orbifold fibration* if the generic fiber is connected; the multiplicity of the fiber over each marked point q_i equals μ_i; and f admits an extension $\bar{f}\colon \overline{X} \to \Sigma_g$ with the same properties.
- Such a map induces an epimorphism $f_\sharp\colon G \twoheadrightarrow \Gamma$, where $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma_{g,s}, \mu)$, and thus a monomorphism $\widehat{f_\sharp}\colon \widehat{\Gamma} \hookrightarrow \widehat{G}$.

THEOREM (ARAPURA + ACM + BW)

$$\mathcal{V}^1(X) = \bigcup_{f \text{ large}} \operatorname{im}(\widehat{f_\sharp}) \cup \bigcup_{f \text{ small}} \left(\operatorname{im}(\widehat{f_\sharp}) \setminus \operatorname{im}(\widehat{f_\sharp})^\circ \right) \cup Z,$$

where Z is a finite set of torsion characters.

THEOREM

Suppose there is a small orbifold fibration $f\colon X \to (\Sigma, \mu)$ and a prime p dividing each μ_i. Then, for all $r \gg 0$ with $p \nmid r$, there is a regular, r-fold cyclic cover $Y \to X$ such that $H_1(Y, \mathbb{Z})$ has non-zero p-torsion.

MULTINETS

DEFINITION (FALK AND YUZVINSKY)

A (k, ℓ)-*multinet* ($k \geq 3$, $\ell \geq 1$) on an arrangement $\mathcal{A}$ consists of:

- A partition $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$.
- An assignment of multiplicities on the hyperplanes, $m \colon \mathcal{A} \to \mathbb{N}$.
- A subset $\mathcal{X} \subseteq L_2(\mathcal{A})$, called the base locus.

Moreover,

- $\sum_{H \in \mathcal{A}_i} m_H = \ell$, for all $i \in [k]$.
- For any hyperplanes H and H' in different classes, $H \cap H' \in \mathcal{X}$.
- For each $X \in \mathcal{X}$, the sum $n_X = \sum_{H \in \mathcal{A}_i \colon H \supset X} m_H$ is independent of i.
- For each $i \in [k]$, the space $\left(\bigcup_{H \in \mathcal{A}_i} H \right) \setminus \mathcal{X}$ is connected.

WLOG, we may assume $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. If all $m_H = 1$, the multinet is *reduced*. If, furthermore, every flat in $\mathcal{X}$ is contained in precisely one hyperplane from each class, this is a *net*.
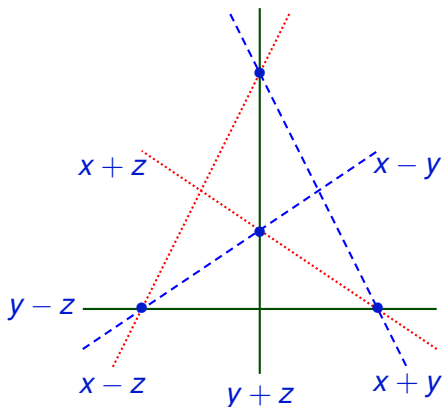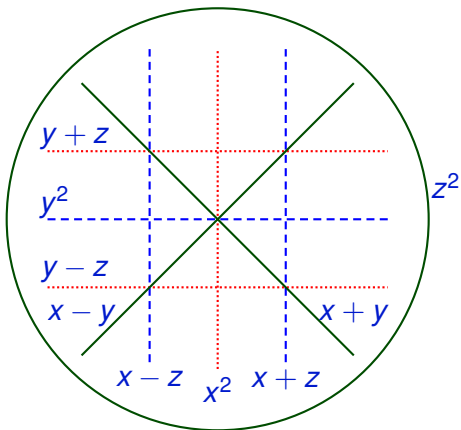
FIGURE : A $(3, 2)$-net on the A_3 arrangement

FIGURE : A $(3, 4)$-multinet on the B_3 arrangement

- Let $\mathcal{A}$ be a (central) arrangement of hyperplanes in $\mathbb{C}^{d+1}$.

- Let $M = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H$, and $U = \mathbb{P}M$. Then $M \cong U \times \mathbb{C}^*$.

- Identify $H_1(M, \mathbb{Z}) = \mathbb{Z}^n$, with basis the meridians $\{x_H \mid H \in \mathcal{A}\}$.

- Let $\mathrm{Hom}(\pi_1(M), \Bbbk^*) = (\Bbbk^*)^n$ be the character torus.

- Then $\mathcal{V}^1(\mathcal{A}) := \mathcal{V}^1(M) \subset (\Bbbk^*)^n$ is isomorphic to $\mathcal{V}^1(U) \subseteq \{t \in (\Bbbk^*)^n \mid t_1 \cdots t_n = 1\} \cong (\Bbbk^*)^{n-1}$.

THEOREM (FALK–YUZ, PEREIRA–YUZ, YUZVINSKY)

Each positive-dimensional, non-local component of $\mathcal{V}^1(\mathcal{A})$ is of the form ρT, where ρ is a torsion character, $T = f^*(H^1(\Sigma_{0,k}, \mathbb{C}^*))$, for some orbifold fibration $f \colon M(\mathcal{A}) \to (\Sigma_{0,k}, \mu)$, and either

- *$k = 2$, and f has at least one multiple fiber, or*

- *$k = 3$ or 4, and f corresponds to a multinet with k classes on the multiarrangement $(\mathcal{A}, m)$, for some m.*

MILNOR FIBERS

- Let $(\mathcal{A}, m)$ be a multi-arrangement with $\gcd\{m_H \mid H \in \mathcal{A}\} = 1$. Set $N = \sum_{H \in \mathcal{A}} m_H$.

- The Milnor fiber $F_m(\mathcal{A})$ is the regular $\mathbb{Z}_N$-cover of $U(\mathcal{A})$ defined by the homomorphism $\delta_m \colon \pi_1(U(\mathcal{A})) \twoheadrightarrow \mathbb{Z}_N$, $x_H \mapsto m_H \bmod N$.

- If $\mathrm{char}(\Bbbk) \nmid N$, then

$$\dim_{\Bbbk} H_q(F_m(\mathcal{A}), \Bbbk) = \sum_{s \geqslant 1} \left| \mathcal{V}_s^q(U(\mathcal{A}), \Bbbk) \cap \mathrm{im}(\widehat{\delta_m}) \right|.$$

- The characteristic polynomial of the algebraic monodromy, $h_* \colon H_q(F_m(\mathcal{A}), \Bbbk) \to H_q(F_m(\mathcal{A}), \Bbbk)$, is given by

$$\Delta_{h,q}^{\Bbbk}(t) = \prod_{s \geqslant 1} \prod_{\substack{\zeta \in \Bbbk^* \colon \zeta^N = 1, \\ \zeta^m \in \mathcal{V}_s^q(U(\mathcal{A}), \Bbbk)}} (t - \zeta).$$

Not every regular, cyclic cover of a projective arrangement complement arises through the Milnor fiber construction. Nevertheless, the Milnor fibers dominate all other cyclic covers.

LEMMA

Let $(\mathcal{A}, m)$ be a multiarrangement, and let $U^{\chi} \to U$ be a regular, r-fold cyclic cover of $U = \mathbb{P}M(\mathcal{A})$, given by $\chi \colon \pi_1(U) \to \mathbb{Z}_r$.

- $U^{\chi} \cong F_m(\mathcal{A})$ if and only if, for some integer k relatively prime to $N = \sum_{H \in \mathcal{A}} m_H$, the least positive integers μ_H satisfying $k \cdot \chi(x_H) = \mu_H \bmod N$ have the property that $\sum_{H \in \mathcal{A}} \mu_H = N$, in which case $\mu = m$.

- There exist infinitely many multiplicity vectors m for which the covering projection $F(\mathcal{A}, m) \to U$ factors through U^{χ}. Moreover, for any prime $p \nmid r$, we may choose m so that the degree of $Q(\mathcal{A}, m)$ is not divisible by p.

DELETION

A *pointed multinet* $(\mathcal{M}, H)$ on an arrangement $\mathcal{A}$ is a multinet structure $\mathcal{M} = ((\mathcal{A}_1, \ldots, \mathcal{A}_k), m, \mathcal{X})$, together with a distinguished hyperplane $H \in \mathcal{A}$ for which $m_H > 1$ and $m_H \mid n_X$ for each $X \in \mathcal{X}$ such that $X \subset H$.

LEMMA

Suppose $\mathcal{A}$ admits a pointed multinet, and set $\mathcal{A}' = \mathcal{A} \backslash \{H\}$. Then $U(\mathcal{A}')$ supports a small pencil, and $\mathcal{V}^1(\mathcal{A}')$ has a component which is a 1-dimensional subtorus, translated by a character of order m_H.

THEOREM

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \backslash \{H\}$ such that $H_1(F_{m'}(\mathcal{A}'), \mathbb{Z})$ has non-zero p-torsion.

POLARIZATION
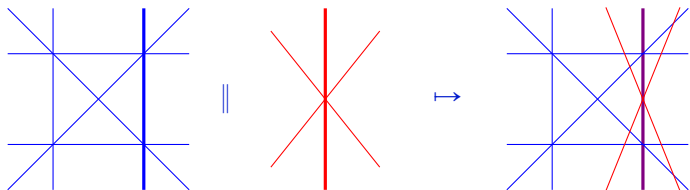
- Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangements in V_1 and V_2, defined by polynomials $Q_1 = \prod_{H \in \mathcal{A}_1} f_H$ and $Q_2 = \prod_{H \in \mathcal{A}_2} g_H$.

- Fix $H_1 \in \mathcal{A}_1$ and $H_2 \in \mathcal{A}_2$. The *parallel connection*, $\mathcal{A}_{1\,H_1} \|_{H_2} \mathcal{A}_2$, is the arrangement in $V_1 \times V_2$ defined by the polynomial

$$f_{H_1} \cdot \prod_{H \in \mathcal{A}_1 \setminus \{H_1\}} f_H \cdot \prod_{H \in \mathcal{A}_2 \setminus \{H_2\}} g_H \in \Bbbk[V_1^*] \otimes_{\Bbbk} \Bbbk[V_2^*] / (f_{H_1} - g_{H_2}).$$

- Falk and Proudfoot (2002): $\mathbb{P}(M_1) \times \mathbb{P}(M_2) \cong \mathbb{P}(M_{1\,H_1} \|_{H_2} M_2)$.

- If $(\mathcal{A}_i, H_i)$ are pointed arrangements and $H \in \mathcal{A}_1$, we let

$$(\mathcal{A}_1, H_1) \circ_H (\mathcal{A}_2, H_2) = (\mathcal{A}_1 \,_H \|_{H_2} \mathcal{A}_2, H_1).$$

- Let $\mathcal{A} = (\{H_1, \ldots, H_n\}, m)$ be a multi-arrangement. The *polarization* $\mathcal{A} \| m$ is the iterated parallel connection

$$\mathcal{A} \circ_{H_1} \mathcal{P}_{m_{H_1}} \circ_{H_2} \cdots \circ_{H_n} \mathcal{P}_{m_{H_n}},$$

where $\mathcal{P}_k$ denotes a pencil of k lines in $\mathbb{C}^2$.

- Note: $\operatorname{rank}(\mathcal{A} \| m) = \operatorname{rank} \mathcal{A} + |\{H \in \mathcal{A} : m_H \geqslant 2\}|$.

- Let $P_k = U(\mathcal{P}_k) \cong \mathbb{C} \backslash \{k - 1 \text{ points}\}$. Write $P(\mathcal{A}) = \prod_{H \in \mathcal{A}} P_{m_H}$. Then

$$U(\mathcal{A}) \times P(\mathcal{A}) \cong U(\mathcal{A} \| m).$$

Polarization construction works well with respect to Milnor fibrations:

$$
\begin{array}{ccc}
F_m(\mathcal{A}) & \longrightarrow & F(\mathcal{A} \parallel m) \\
\downarrow & & \downarrow \\
U(\mathcal{A}) & \longrightarrow & U(\mathcal{A} \parallel m),
\end{array}
$$

THEOREM

Suppose $\mathcal{A}$ admits a pointed multinet, with distinguished hyperplane H and multiplicity m. Let p be a prime dividing m_H. There is then a choice of multiplicities m' on the deletion $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ such that $H_q(F(\mathcal{B}), \mathbb{Z})$ has p-torsion, where $\mathcal{B} = \mathcal{A}' \parallel m'$ and $q = 1 + |\{K \in \mathcal{A}' : m'_K \geqslant 3\}|$.

EXAMPLE

- Let $\mathcal{A}$ be reflection arrangement of type B_3, with
  $Q(\mathcal{A}) = xyz(x-y)(x+y)(x-z)(x+z)(y-z)(y+z)$.

- Recall $\mathcal{A}$ supports a $(3, 4)$-multinet.

- Let $\mathcal{A}' = \mathcal{A}\backslash\{z = 0\}$ be the deleted B_3 arrangement.

- Pick $m' = (8, 1, 3, 3, 5, 5, 1, 1)$.

- Let $\mathcal{B} = \mathcal{A}'\|m'$, an arrangement of 27 hyperplanes in $\mathbb{C}^8$

- Then $H_6(F(\mathcal{B}), \mathbb{Z})$ has 2-torsion of rank 108.

- $\Delta_6^{\mathbb{C}}(t) = (t-1)^{11968}$, yet $\Delta_6^{\overline{\mathbb{F}_2}}(t) = (t-1)^{11968}(t^2 + t + 1)^{54}$.

REFERENCES

📄 G. Denham, A. Suciu, *Multinets, parallel connections, and Milnor fibrations of arrangements*, arxiv:1209.3414.

📄 A. Suciu, *Characteristic varieties and Betti numbers of free abelian covers*, arxiv:1111.5803, to appear in IMRN, 2013.

📄 A. Suciu, *Hyperplane arrangements and Milnor fibrations*, arxiv:1301.4851.