Lower central series and free resolutions of arrangements

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Lower central series

G finitely-generated group.

- LCS: $G = G_1 \geq G_2 \geq \cdots,\ G_{k+1} = [G_k, G]$
- LCS quotients: $\operatorname{gr}_k G = G_k/G_{k+1}$
- LCS ranks: $\phi_k(G) = \operatorname{rank}(\operatorname{gr}_k G)$

Hyperplane arrangements

$\mathcal{A} = \{H_1, \dots, H_n\}$ set of hyperplanes in $\mathbb{C}^\ell$.

- Intersection lattice: $L(\mathcal{A}) = \left\{\bigcap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\right\}$
- Complement: $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H$

Many topological invariants of $M = M(\mathcal{A})$ are determined by the combinatorics of $L(\mathcal{A})$. E.g.:

- Cohomology ring:

  $A := H^*(M, \mathbb{Q}) = E/I$ (Orlik-Solomon algebra)
- Betti numbers and Poincaré polynomial:

  $b_i := \dim H_i(M, \mathbb{Q}) = \sum_{X\in L_i(\mathcal{A})} (-1)^i \mu(X)$

  $P(M, t) := \operatorname{Hilb}(A, t) = \sum_{i=0}^{\ell} b_i t^i$

$G = \pi_1(M)$ is not always combinatorially determined. Nevertheless, its LCS ranks are determined by $L(\mathcal{A})$.

Problem. Find an *explicit* combinatorial formula for the LCS ranks, $\phi_k(G)$, of an arrangement group G (at least for certain classes of arrangements).

LCS formulas

- **Witt formula**

  $\mathcal{A} = \{n \text{ points in } \mathbb{C}\}$

  $G = F_n$ (free group on n generators)

  $\phi_k(F_n) = \frac{1}{k}\sum_{d|k}\mu(d)n^{k/d}$, or:

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = 1 - nt$$

- **Kohno** [1985]

  $\mathcal{B}_\ell = \{z_i - z_j = 0\}_{1\le i<j\le \ell}$ braid arrangement in $\mathbb{C}^\ell$

  $G = P_\ell$ (pure braid group on ℓ strings)

$$\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = \prod_{j=1}^{\ell-1}(1-jt)$$

- **Falk-Randell LCS formula** [1985]

  If $\mathcal{A}$ fiber-type $\quad [\Longleftrightarrow L(\mathcal{A})$ supersolvable (Terao)]

  with exponents $d_1, \ldots, d_\ell$

  $G = F_{d_\ell} \rtimes \cdots \rtimes F_{d_2} \rtimes F_{d_1}$

  $\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$

  and so:

$$\boxed{\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = P(M,-t)}$$

- **Shelton-Yuzvinsky** [1997], **Papadima-Yuz** [99]

  If $\mathcal{A}$ Koszul (i.e., $A = H^*(M,\mathbb{Q})$ is a Koszul algebra) then the LCS formula holds.

Remark. There are many arrangements for which the LCS formula fails. In fact, as noted by Peeva, there are arrangements for which

$$\prod_{k\geq 1}(1-t^k)^{\phi_k} \neq \mathrm{Hilb}(N,-t),$$

for any graded-commutative algebra N.

LCS and free resolutions

We want to reduce the problem of computing $\phi_k(G)$ to that of computing the graded Betti numbers of certain free resolutions involving the OS-algebra $A = E/I$.

The starting point is the following (known) formula:

$$\boxed{\prod_{k=1}^{\infty}(1-t^k)^{-\phi_k} = \sum_{i=0}^{\infty} b_{ii}\, t^i}$$

where $b_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^A(\mathbb{Q},\mathbb{Q})_j$ is the i^{th} Betti number (in degree j) of a minimal free resolution of $\mathbb{Q}$ over A:

$$\cdots \longrightarrow \oplus_j A^{b_{2j}}(-j) \longrightarrow A^{b_1}(-1) \xrightarrow{(e_1 \ \cdots \ e_{b_1})} A \longrightarrow \mathbb{Q} \to 0$$

Betti diagram:

$$\begin{array}{rccccccl} 0: & 1 & b_1 & b_{22} & b_{33} & \dots & \leftarrow \text{linear strand} \\ 1: & . & . & b_{23} & b_{34} & \dots & \\ 2: & . & . & b_{24} & b_{35} & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{array}$$

Formula follows from:

- Sullivan: M formal $\Longrightarrow$ assoc. graded Lie algebra of $G = \pi_1(M) \cong$ holonomy Lie algebra of $H_* = H_*(M, \mathbb{Q})$:

$$\operatorname{gr} G := \bigoplus_{k\geq 1} G_k/G_{k+1} \otimes \mathbb{Q} \cong \mathfrak{g} := \mathbb{L}(H_1)/\operatorname{im}(\nabla\colon H_2 \to H_1 \wedge H_1)$$

- Poincaré-Birkhoff-Witt:

$$\prod_{k=1}^{\infty} (1-t^k)^{-\phi_k} = \operatorname{Hilb}(U(\mathfrak{g}), t)$$

- Shelton-Yuzvinsky:

$$U(\mathfrak{g}) = \overline{A}^{!}$$

- Priddy, Löfwall:

$$\overline{A}^{!} \cong \bigoplus_{i\geq 0} \operatorname{Ext}^i_A(\mathbb{Q}, \mathbb{Q})_i$$

Here $\overline{A} = E/I[2]$ is the quadratic closure of A, and $\overline{A}^{!}$ is its Koszul dual.

Remark. If A is a Koszul algebra, i.e.,

$$\operatorname{Ext}^i_A(\mathbb{Q}, \mathbb{Q})_j = 0, \quad \text{for } i \neq j,$$

then $A = \overline{A}$ and $\operatorname{Hilb}(A^!, t) \cdot \operatorname{Hilb}(A, -t) = 1$. This yields the LCS formula of Shelton-Yuzvinsky.

Change of rings spectral sequence

The idea now is to further reduce the computation to that of a (minimal) free resolution of A over E,

$$\cdots \longrightarrow \oplus_j E^{b'_{2j}}(-j) \longrightarrow \oplus_j E^{b'_{1j}}(-j) \longrightarrow E \longrightarrow A \to 0$$

and its Betti numbers, $b'_{ij} = \dim_{\mathbb{Q}} \operatorname{Tor}_i^E(A, \mathbb{Q})_j$.

This problem (posed by Eisenbud-Popescu-Yuzvinsky [1999]) is interesting in its own right.

Let

$$a_j = \#\{\text{minimal generators of } I \text{ in degree } j\}$$

Clearly, $a_2 + b_2 = \binom{b_1}{2}$. A 5-term exact sequence argument yields:

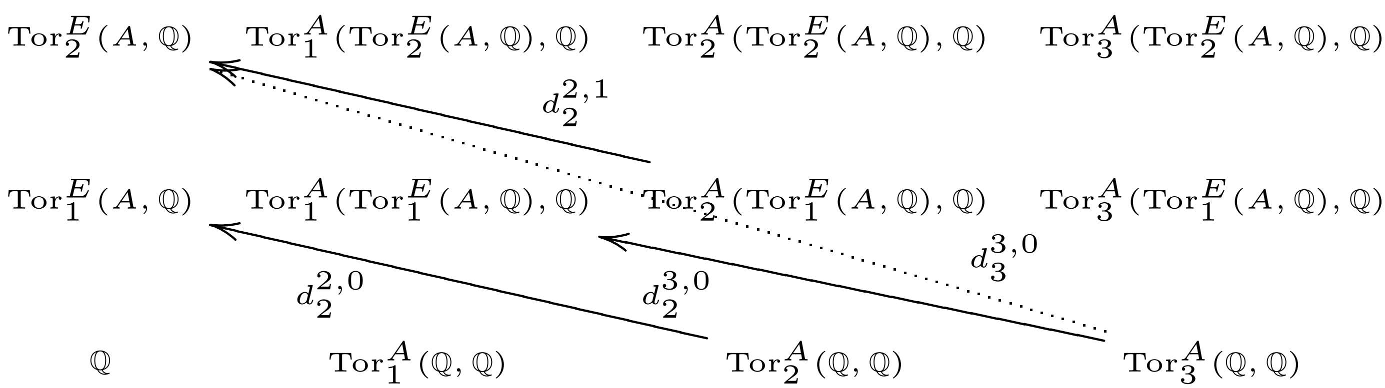
Lemma. *$a_j = b'_{1j} = b_{2j}$, for all $j > 2$.*

Betti diagram:

$$\begin{array}{rccccl}
0: & 1 & . & . & . & \\
1: & . & a_2 & b'_{23} & b'_{34} & \ldots \quad \leftarrow \text{linear strand} \\
2: & . & a_3 & b'_{24} & b'_{35} & \ldots \\
\\
\ell-1: & . & a_\ell & b'_{2,\ell+1} & b'_{3,\ell+2} & \ldots \\
\ell: & . & . & . & . &
\end{array}$$

Key tool: Cartan-Eilenberg change-of-rings spectral sequence associated to the ring maps $E \twoheadrightarrow A \twoheadrightarrow \mathbb{Q}$:

$$\boxed{\mathrm{Tor}_i^A\left(\mathrm{Tor}_j^E(A,\mathbb{Q}),\mathbb{Q}\right) \Longrightarrow \mathrm{Tor}_{i+j}^E(\mathbb{Q},\mathbb{Q})}$$



The (Koszul) resolution of $\mathbb{Q}$ as a module over E is linear, with

$$\dim \mathrm{Tor}_i^E(\mathbb{Q},\mathbb{Q})_i = \binom{n+i-1}{i}$$

Thus, if we know $\mathrm{Tor}_i^E(A,\mathbb{Q})$, we can find $\mathrm{Tor}_i^A(\mathbb{Q},\mathbb{Q})$, provided we can compute the differentials $d_r^{p,q}$.

We carry out this program, at least in low degrees. As a result, we express ϕ_k, $k \le 4$, solely in terms of the resolution of A over E.

Theorem. *For an arrangement of n hyperplanes:*

$$\phi_1 = n$$

$$\phi_2 = a_2$$

$$\phi_3 = b'_{23}$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4$$

where

$$a_2 = \#\{\textit{generators of } I_2\} = \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X)}{2}$$

$$b'_{23} = \#\{\textit{linear first syzygies on } I_2\}$$

$$b'_{34} = \#\{\textit{linear second syzygies on } I_2\}$$

$$\delta_4 = \#\{\textit{minimal, quadratic, Koszul syzygies on } I_2\}$$

ϕ_1, ϕ_2: elementary

ϕ_3: recovers a formula of Falk [1988]

ϕ_4: new

Decomposable arrangements

Let $\mathcal{A}$ be an arrangement of n hyperplanes.

Recall that $\phi_1 = n$, $\phi_2 = \sum_{X \in L_2(\mathcal{A})} \phi_2(F_{\mu(X)})$

Falk [1989]:

$$\boxed{\phi_k \geq \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})} \quad \text{for all } k \geq 3 \qquad (*)$$

If the lower bound is attained for $k = 3$, we say that $\mathcal{A}$ is *decomposable* (or *local*, or *minimal linear strand*).

Conjecture (MLS LCS). If $\mathcal{A}$ decomposable, equality holds in (*), and so

$$\boxed{\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - t)^n \prod_{X \in L_2(\mathcal{A})} \frac{1 - \mu(X)t}{1 - t}}$$

Proposition. *The conjecture is true for* $k = 4$*:*

$$\phi_4 = \tfrac{1}{4} \sum_{X \in L_2(\mathcal{A})} \mu(X)^2(\mu(X)^2 - 1)$$

If $\mathcal{A}$ decomposable, we compute the entire linear strand of the resolution of A over E. If, moreover, $\operatorname{rank}\mathcal{A} = 3$, we compute all b'_{ij} from Möbius function.

Example. $\mathcal{A} = \{H_0, H_1, H_2\}$ pencil of 3 lines in $\mathbb{C}^2$. OS-ideal generated by $\partial e_{012} = (e_1 - e_2) \wedge (e_0 - e_2)$. Minimal free resolution of A over E:

$$0 \leftarrow A \longleftarrow E \xleftarrow{(\partial e_{012})} E(-1) \xleftarrow{(e_1 - e_2 \;\; e_0 - e_2)} E^2(-2)$$
$$\xleftarrow{\begin{pmatrix} e_1 - e_2 & e_0 - e_2 & 0 \\ 0 & e_1 - e_2 & e_0 - e_2 \end{pmatrix}} E^3(-3) \longleftarrow \cdots$$

Thus, $b'_{i,i+1} = i$, for $i \geq 1$, and $b'_{i,i+r} = 0$, for $r > 1$.

Lemma. *For any arrangement $\mathcal{A}$:*

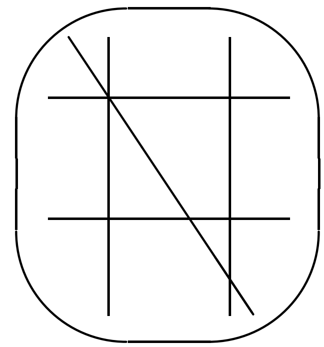
$$b'_{i,i+1} \geq i \sum_{X \in L_2(\mathcal{A})} \binom{\mu(X) + i - 1}{i+1}$$

$$\delta_4 \leq \sum_{(X,Y) \in \binom{L_2(\mathcal{A})}{2}} \binom{\mu(X)}{2}\binom{\mu(Y)}{2}.$$

If $\mathcal{A}$ is decomposable, then equalities hold.

Lemma + Theorem $\Longrightarrow$ Proposition.

Example. X_3 arrangement (smallest non-LCS)



```
res of residue field over OS alg
total: 1 6 25 92 325 1138
    0: 1 6 24 80 240 672
    1: . . 1  12 84  448
    2: . . .  .   1   18
```

```
res of OS alg over exterior algebra
total: 1 4 15 42 97 195 354 595 942 1422 2065
    0: 1 .  .  .  .   .   .   .   .    .    .
    1: . 3  6  9 12  15  18  21  24   27   30
    2: . 1  9 33 85 180 336 574 918 1395 2035
```

We find: $b'_{i,i+1} = 3i$, $b'_{i,i+2} = \frac{1}{8} i(i+1)(i^2+5i-2)$.
Thus:

$$\phi_1 = n = 6$$

$$\phi_2 = a_2 = 3$$

$$\phi_3 = b'_{23} = 6$$

$$\phi_4 = \binom{a_2}{2} + b'_{34} - \delta_4 = 3 + 9 - 3 = 9$$

Conjecture says: $\prod_{k=1}^{\infty}(1-t^k)^{\phi_k} = (1-2t)^3$

i.e.: $\phi_k(G) = \phi_k(F_2^3)$, though definitely $G \not\cong F_2^3$.

Graphic arrangements

$\mathsf{G} = (\mathcal{V}, \mathcal{E})$ subgraph of the complete graph K_ℓ. (Assume no isolated vertices, so that $\mathcal{E}$ determines G.)

The *graphic arrangement* (in $\mathbb{C}^\ell$) associated to G:

$$\mathcal{A}_\mathsf{G} = \{\ker(z_i - z_j) \mid \{i, j\} \in \mathcal{E}\}$$

- $\mathsf{G} = K_\ell \implies \mathcal{A}_\mathsf{G}$ braid arrangement
- $\mathsf{G} = \mathrm{A}_{\ell+1}$ diagram $\implies \mathcal{A}_\mathsf{G}$ Boolean arrangement
- $\mathsf{G} = \ell$-gon $\implies \mathcal{A}_\mathsf{G}$ generic arrangement

Theorem. (Stanley, Fulkerson-Gross) $\mathcal{A}_\mathsf{G}$ *is supersolvable* $\iff \forall$ *cycle in* G *of length* > 3 *has a chord.*

Lemma. (Cordovil-Forge [2001], S-S)

$$a_j = \#\{\textit{chordless } j+1 \textit{ cycles}\}$$

Together with a previous lemma ($a_j = b_{2j}$), this gives:

Corollary. $\mathcal{A}_\mathsf{G}$ *supersolvable* $\iff \mathcal{A}_\mathsf{G}$ *Koszul.*

For arbitrary $\mathcal{A}$: $\implies$ true (Shelton-Yuzvinsky)

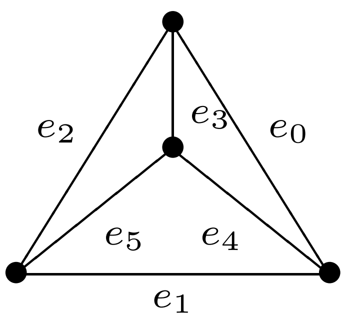
$\Longleftarrow$ open problem

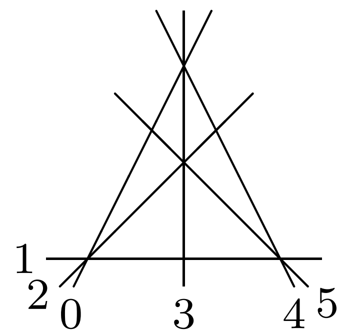
Proposition. *For a graphic arrangement:*

$$b'_{i,i+1} = i(\kappa_3 + \kappa_4)$$

$$\delta_4 \le \binom{\kappa_3}{2} - 6(\kappa_4 + \kappa_5)$$

Example. Braid arrangement $\mathcal{B}_4 = \mathcal{A}_{K_4}$





Free resolution of A over E:

$$0 \leftarrow A \longleftarrow E \xleftarrow{\partial_1} E^4(-2) \xleftarrow{\partial_2} E^{10}(-3) \longleftarrow \cdots$$

$$\partial_1 = \begin{pmatrix} \partial e_{145} & \partial e_{235} & \partial e_{034} & \partial e_{012} \end{pmatrix}$$

$$\partial_2 = \begin{pmatrix} e_1-e_4 & e_1-e_5 & 0 & 0 & 0 & 0 & 0 & 0 & e_3-e_0 & e_2-e_0 \\ 0 & 0 & e_2-e_3 & e_2-e_5 & 0 & 0 & 0 & 0 & e_0-e_1 & e_0-e_4 \\ 0 & 0 & 0 & 0 & e_0-e_3 & e_0-e_4 & 0 & 0 & e_1-e_5 & e_2-e_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & e_0-e_1 & e_0-e_2 & e_3-e_5 & e_4-e_5 \end{pmatrix}$$

The 2 non-local linear syzygies	$\longleftrightarrow$	2-dim essential component in the resonance variety $R_1(\mathcal{B}_4)$

Get: $b'_{i,i+1} = 5i$, $\delta_4 = 0$.

For a graph G, let

$$\kappa_s = \#\{\text{complete subgraphs on } s \text{ vertices}\}$$

From the Theorem, and the Proposition above, we get:

Corollary. *The LCS ranks of $\mathcal{A}_{\mathsf{G}}$ satisfy:*

$$\begin{aligned} \phi_1 &= \kappa_2 \\ \phi_2 &= \kappa_3 \\ \phi_3 &= 2(\kappa_3 + \kappa_4) \\ \phi_4 &\geq 3(\kappa_3 + 3\kappa_4 + 2\kappa_5) \end{aligned}$$

Moreover, if $\kappa_4 = 0$, equality holds for ϕ_4.

ϕ_3: answers a question of Falk.

Conjecture (Graphic LCS).

$$\phi_k = \tfrac{1}{k} \sum_{d|k} \sum_{j=2}^{\kappa_2 - 1} \sum_{s=j}^{\kappa_2 - 1} (-1)^{s-j} \binom{s}{j} \kappa_{s+1}\, \mu(d)\, j^{\frac{k}{d}}$$

or

$$\boxed{\prod_{k=1}^{\infty} \left(1 - t^k\right)^{\phi_k} = \prod_{j=1}^{\kappa_2 - 1} (1 - jt)^{\sum_{s=j}^{\kappa_2 - 1} (-1)^{s-j} \binom{s}{j} \kappa_{s+1}}}$$